AP Calculus AB - Notes

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Introduction

Calculus is defined as the mathematical study of continuous change. AP Calculus AB is designed to mimic a first-semester college course devoted to certain topics in differential and integral calculus. Our studies use prior knowledge of *algebra, geometry, trigonometry, analytic geometry, and elementary functions*. Students should already be familiar with the aforementioned topics and related notation from prerequisite classes.

Also be aware that these notes are not a textbook. Use these to supplement your in-class learning and note-taking! Each section includes examples and exercises; take time to work through much of them to solidify your understanding of each topic.

Calculus is the capstone mathematics subject in high school and early undergraduate college—all prerequisite courses in mathematics culminate in calculus. Why is it so important? Calculus provides a mathematical foundation for nearly every career field in STEM:

Engineering: All branches of engineering, including mechanical, electrical, aerospace, etc. rely on calculus to model and analyze problems relating to motion, forces, thermodynamics, and more.

Computer Science: Calculus is used in various areas of computer science, particularly in algorithms, computational geometry, computer graphics, machine learning, and artificial intelligence. Understanding calculus helps in designing efficient algorithms and analyzing their performance.

Finance and Economics: Calculus is utilized in economic modeling, optimization of utility functions, and analyzing financial derivatives. It is important for understanding concepts like marginal utility, optimization of profit or utility functions, and pricing of financial instruments.

Biology and Chemistry: Calculus is used in mathematical modeling of biological processes, such as population dynamics, enzyme kinetics, and more. In chemistry, calculus is applied in understanding, for instance, reaction rates, chemical equilibrium, and thermodynamics.

The list goes on *much* longer! If any students have aspirations to pursue a career in STEM (which many of you are), then it is imperative that you get comfortable with many of these topics.

A graphing calculator is required for this course. My personal recommendation is the Texas Instruments TI-84 series.

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1 Limits and Continuity

1.1 Defining Limits

We begin by developing a little notation. The following definition (and many in this chapter) is informal, but sufficient for this course.

Definition 1.1. As x approaches a number a from the left and right and $f(x)$ approaches a number L, we write

$$\lim_{x \to a} f(x) = L,$$

which reads: 'the *limit* as x approaches a of $f(x)$ is L'.

Even more informally, think: 'as the horizontal component of the graph, x, gets really close to a, the vertical component of the graph, $f(x)$ (the y-value of the function) gets really close to L'.

It's important to note that when we're talking about the limit of f as $x \to a$, x does not actually reach a. Whether $f(a)$ exists is not necessary for the definition of a limit. What's important is how the function behaves as x gets *really close* to a.

For example, we should be comfortable with the idea of 'holes' in a function from a prior algebra class.

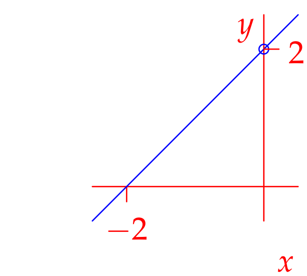
Example 1.2. Consider the function $f(x) = \frac{x^2+2x}{x}$. If we wish to find the limit as $x \to 0$, it is tempting to simply substitute $x = 0$ into $f(x)$. However, you'll notice we would get the value $\frac{0}{0}$, which is meaningless. With some algebra,

$$f(x) = \frac{x^2 + 2x}{x} = \frac{x(x+2)}{x} = x + 2, \quad x \neq 0.$$

The graph of $y = f(x)$ is the straight line $y = x + 2$ with the point $(0, 2)$ missing, called a hole. [1]

However, even though this point is missing, the *limit* of $f(x)$ as x approaches 0 does indeed exist. In particular, as $x \to 0$ from both the left side and right side, $f(x) \to 2$. Notice visually that as x gets close to 0, the y-value of the graph seems to approach 2.

We write $\lim_{x \to 0} \frac{x^2 + 2x}{x} = 2$.



$f(x)$ has missing point $(0, 2)$

In practice we don't usually need to graph functions each time to evaluate limits. Most of the time, they can be calculated algebraically. We'll start with rational functions. The basic method for finding limits of rational functions $f(x)$ as $x \to a$ is:

If possible, simply substitute a into $f(x)$.

If not, simplify $f(x)$ first, then substitute a into the simplified expression.

[1] More formally, a *point of discontinuity.* We will return to this definition later.

And, in general, if a function is 'nice', substituting a into $f(x)$ will provide the correct limit.

Examples 1.3. 1. To find $\lim_{x\to 4} x^2$, notice that if we simply substitute $x = 4$ into x^2, we get no 'problem'.

$$\implies \lim_{x\to 4} x^2 = 4^2 = 16.$$

2. $$\lim_{x\to -5} \frac{x^2+6x+5}{x^2-2x-35} = \lim_{x\to -5} \frac{(x+5)(x+1)}{(x+5)(x-7)} = \lim_{x\to -5} \frac{x+1}{x-7} = \frac{-5+1}{-5-7} = \frac{1}{3}$$

One-sided Limits

Sometimes, we may want to observe the behavior of a function as $x \to a$ from only one side: either from the right or only on the left.

Definition 1.4. As x approaches a number a, where $x > a$ and $f(x)$ approaches a number L, we write

$$\lim_{x\to a^+} f(x) = L,$$

which reads: 'the limit as x approaches a from above (the right) is L'.
Similarly, we have the limit from below (the left): $\lim_{x\to a^-} f(x) = L$

This can be informally thought: 'as x gets really close to a from the left- or right-side, $f(x)$ (the y-value of the function) gets really close to L'.

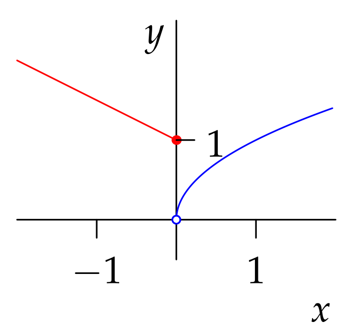
Theorem 1.5. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.
Equivalently, we say that the limit does not exist if and only if the latter equality is false.

Theorem 1.5 says that the (overall) limit of a function exists at an x-value only if the left- and right-side limits agree with each other, and vice versa. Otherwise, the limit of the function *does not exist*, and we commonly write

$$\lim_{x\to a} f(x) = \text{DNE}$$

Examples 1.6. 1. Consider the piecewise-defined function

$$f(x) = \begin{cases} -\frac{1}{2}x + 1, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$$



Notice that if we trace the function from the left as $x \to 0^-$, the y-value of the function seems to approach 1, and if we trace the function from the right as $x \to 0^+$, the y-value of the function seems to approach 0. The value $f(0) = 1$ is not relevant to the value of the limit.

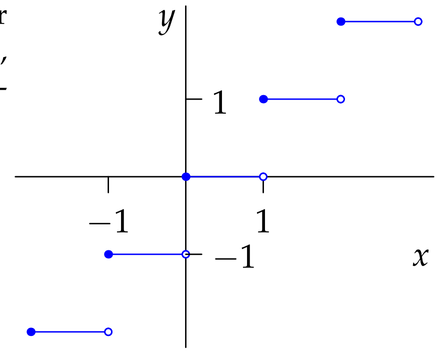
$$\implies \lim_{x\to 0^-} f(x) = 1 \neq 0 = \lim_{x\to 0^+} f(x) \implies \lim_{x\to 0} f(x) = \text{DNE}$$

2. The *step function*, also known as the *greatest integer function* or *floor function*, denoted usually as $g(x) = [[x]]$ or $g(x) = \lfloor x \rfloor$, returns the greatest integer less than or equal to itself. For instance,

   $$g(3.2) = \lfloor 3.2 \rfloor = 3$$
   $$g(-5) = \lfloor -5 \rfloor = -5$$
   $$g(8.99) = \lfloor 8.99 \rfloor = 8$$

   Observe that as $x \to 2^-$, $g(x) \to 1$, and as $x \to 2^+$, $g(x) \to 2$.

   $$\implies \lim_{x \to 2^-} g(x) = 1 \neq 2 = \lim_{x \to 2^+} g(x) \implies \lim_{x \to 2} g(x) = \text{DNE}$$

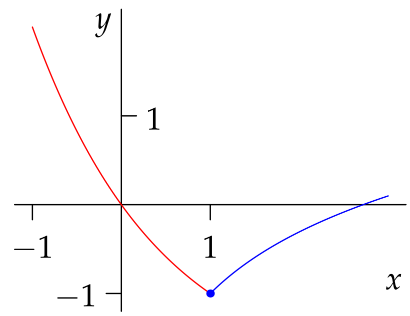
   

3. Consider the function h defined by

   $$h(x) = \begin{cases} 2^{1-x} - 2, & x < 1 \\ \ln x - 1, & x \geq 1 \end{cases}$$

   As $x \to 1^-$, $h(x) \to -1$. Similarly, as $x \to 1^+$, $h(x) \to -1$.

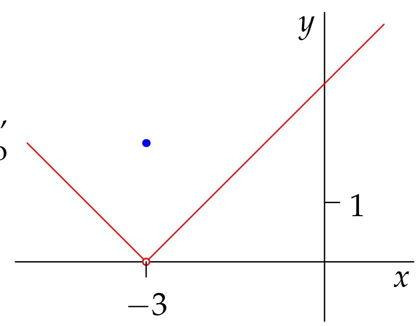
   $$\implies \lim_{x \to 1^-} h(x) = \lim_{x \to 1^+} h(x) = -1 \implies \lim_{x \to 1} h(x) = -1$$

4. Let $r(x) = \begin{cases} |x+3|, & x \neq -3 \\ 2, & x = -3 \end{cases}$

   Notice that $r(-3) = 2$. However, this value is irrelevant; again, we only care about how the function behaves as it gets *close* to -3. And as $x \to -3$ from the left- and right-side, $r(x) \to 0$.

   $$\implies \lim_{x \to -3} r(x) = 0$$

One-sided limits are most interesting at the 'breakpoints' of a piecewise function. And as we see in example 3 and 4 of 1.6, if the two 'branches' of the graph of a piecewise function connect at a point (regardless of whether the function is defined at that point), then the limit of the function as x approaches that value exists.

Infinite Limits and Asymptotes

Recall curves that have *asymptotes*: a line such that the distance between the curve and the line approaches zero as one or both of the x- or y-coordinates approaches $\pm\infty$. Otherwise said, the curve gets very close to the asymptote but does not intersect it at any finite distance.

Definition 1.7. Suppose that the values of $f(x)$ get arbitrarily large as x approaches a: we write[2]

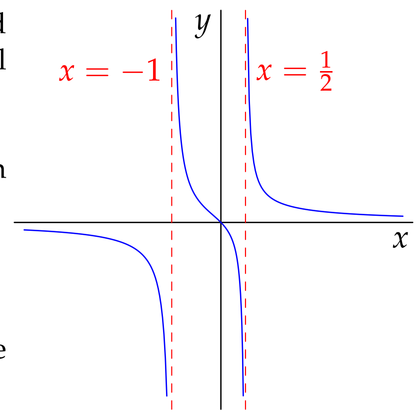
$$\lim_{x \to a} f(x) = \infty$$

The negative limits $\lim_{x \to a} f(x) = -\infty$ and the one-sided limits $\lim_{x \to a^{\pm}} f(x) = \pm\infty$ are similar. If any of these limits are $\pm\infty$, we say that f has a *vertical asymptote* at $x = a$.

In the context of Definition 1.7, we are using limits to define vertical asymptotes, rather than the other way around.

We are saying that if the y-value of a function f goes up or down forever at an x-value a from either the left- or right-side, then f has a vertical asymptote at $x = a$. Try to visualize this!

Example 1.8. $f(x) = \frac{x}{2x^2+x-1}$ has vertical asymptotes at $x = \frac{1}{2}$ and $x = -1$. Remember how to find the vertical asymptotes of a rational function (factorize the denominator!).



Notice that as $x \to -1^-$, the y-value of the graph goes down forever ($f(x) \to -\infty$), and as $x \to -1^+$, $f(x) \to \infty$.

$$\implies \lim_{x \to -1^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -1^-} f(x) = \infty$$

So f has a vertical asymptote at $x = -1$ by Definition 1.7. The same argument can be used to identify the vertical asymptote at $x = \frac{1}{2}$.

We can also use limits to define horizontal asymptotes.

Definition 1.9. Suppose that the values of $f(x)$ get arbitrarily close to L as x becomes sufficiently large. We say f has a *limit at infinity* and we write:

$$\lim_{x \to \infty} f(x) = L$$

The *limit at negative infinity* $\lim_{x \to -\infty} f(x) = L$ is similar. If the values $f(x)$ get arbitrarily close to some **finite** value value L as $x \to \infty$ or $x \to -\infty$, then the line $y = L$ is a *horizontal asymptote* of f.

Again, in the context of Definition 1.9, we use limits to define horizontal asymptotes.

Colloquially, if the y-value of a graph gets *really close* to a number L as x goes very far to the right/left ($x \to \pm\infty$), then we say that the limit of $f(x)$ as x approaches positive or negative infinity is L.

And since the curve doesn't actually reach L, $f(x)$ has a horizontal asymptote at the line $y = L$.

Notice the emphasis on **finite** in the definition. If $f(x) \to \pm\infty$ as $x \to \pm\infty$, then $y = \pm\infty$ is *not* a horizontal asymptote of f!

Horizontal asymptotes and limits at infinity are usually (but not always) presented using rational functions. Finding limits at infinity can be done graphically, but that's obviously tedious; let's play around with them algebraically first.

[2]This depends on the author; some would say that if $f(x) \to \pm\infty$, then $\lim f(x) = \text{DNE}$, since $f(x)$ does not approach a real number. In AP, we use the notation given in 1.7.

Let

$$f(x) = \frac{Ax^n + Bx^{n-1} + \dots}{Cx^m + Dx^{m-1} + \dots}$$

represent any arbitrary rational function with degree (i.e. highest power) n in the numerator and degree m in the denominator. We wish to find

$$\lim_{x\to\infty} \frac{Ax^n + Bx^{n-1} + \dots}{Cx^m + Dx^{m-1} + \dots}$$

For rational functions, the *leading terms* in the numerator and denominator determine the behavior of the function at very large (positive or negative) values of x (Think about 100^3 versus 100^2 for example. The former is *much* bigger than the latter. This is inflated for even larger values of x). So we can discard the non-leading terms in the limit at infinity:

$$\lim_{x\to\infty} \frac{Ax^n + Bx^{n-1} + \dots}{Cx^m + Dx^{m-1} + \dots} = \lim_{x\to\infty} \frac{Ax^n}{Cx^m} = \dots$$

From here, we have three cases:

Case 1: if $n < m$, then we may cancel out like x terms in the numerator and denominator, until x^k for some integer k remains in the denominator.

$$= \lim_{x\to\infty} \frac{A}{Cx^k}$$

Now imagine 'plugging in'[3] ∞ to evaluate the limit. A fraction with a finite numerator and an infinitely large denominator will be close to 0, so $\lim f(x) = 0$. Consequently, f has a horizontal asymptote at $y = 0$.

Case 2: if $n = m$, then the powers of x in the numerator and denominator are equal. Cancel them out, and we are left with

$$= \lim_{x\to\infty} \frac{A}{C} = \frac{A}{C}$$

The limit is $\frac{A}{C}$, the ratio of the leading coefficient in the numerator to the leading coefficient in the denominator $\implies f$ has a horizontal asymptote at $y = \frac{A}{C}$.

Case 3: if $n > m$, then cancel similarly to the other two cases until x^k remains in the numerator.

$$= \lim_{x\to\infty} \frac{Ax^k}{C}$$

When we 'plug in' ∞ or $-\infty$, the numerator will grow arbitrarily large $\implies \lim f(x) = \infty$ or $-\infty$, depending on the function.

[3] This is an abuse of language, but acceptable for AP Calculus.

Examples 1.10. 1. $\lim_{x\to\infty} \frac{2x^2+3x+1}{6-7x^2-6x} = \lim_{x\to\infty} \frac{2x^2}{-7x^2} = \lim_{x\to\infty} \frac{2}{-7} = -\frac{2}{7}$

Since this limit at infinity is a finite number, this rational function has a horizontal asymptote at $y = -\frac{2}{7}$.
It also has two vertical asymptotes. They can be found by using the quadratic formula on the denominator and finding its zeroes (try it!).

2. We find $\lim_{x\to-\infty} \frac{4}{\sqrt{5+3x^7}}$. Even though the term $3x^7$ is enclosed in a square root, we can use the same methods previously discussed. Notice that we can equate $\sqrt{3x^7}$ to $\sqrt{3}x^{\frac{7}{2}}$:

$$\lim_{x\to-\infty} \frac{4}{\sqrt{5+3x^7}} = \lim_{x\to-\infty} \frac{4}{\sqrt{3}x^{\frac{7}{2}}} = \frac{4}{-\infty} = 0$$

The function has a horizontal asymptote at $y = 0$.

3. $\lim_{x\to-\infty} \frac{10-x^4}{3x+2x^2} = \lim_{x\to-\infty} \frac{-x^4}{2x^2} = \lim_{x\to-\infty} -\frac{1}{2}x^2 = -\frac{1}{2}(-\infty)^2 = -\frac{1}{2}(+\infty) = \frac{1}{2}(-\infty) = -\infty$

Because its limits at infinity are not finite, this rational function does not have any horizontal asymptotes.

Limit Laws

Theorem 1.11. Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, and that c is constant.
Then the following limits exist and can be evaluated as follows:

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
4. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x)$, provided $\lim_{x\to a} g(x) \neq 0$

Theorem 1.11 essentially says that any 'nice' combination of functions has exactly the limits you'd expect.
The Theorem also holds for one-sided limits, and with some care, [4]for infinite limits. For example, if $\lim_{x\to 4} f(x) = -5$ and $\lim_{x\to 4} g(x) = -\infty$, then

$$\lim_{x\to 4} f(x)g(x) = \infty \quad \text{and} \quad \lim_{x\to a}[f(x) + g(x)] = -\infty$$

[4]If you end up with an *indeterminate form* $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, etc., then the rules don't apply. We will deal with these limits later using L'Hôpital's Rule.

Exercises 1.1. 1. Selected values for a function g are shown in the table below.

x	1	1.5	1.8	1.9	1.99	2.01	2.1	2.2	2.5	3
$g(x)$	0	-2.5	-11.2	-26.1	-296.01	304.01	34.1	8.55	10.5	8

Determine $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$. Does $\lim_{x\to 2} g(x)$ exist? Why or why not?

2. Compute the following limits, if they exist.

(a) $\lim_{x\to -1} (2x^2 - 15x)$ (b) $\lim_{x\to 6} 23$ (c) $\lim_{x\to -3} \frac{27 - x^3}{x^2 - 9}$

(d) $\lim_{x\to -2} \frac{x^3 + 5x^2 - 4x - 20}{3x^2 + 2x - 8}$ (e) $\lim_{\theta\to \frac{\pi}{3}} \sec\theta$ (f) $\lim_{x\to \frac{3}{2}} \lfloor x \rfloor$

3. Find the vertical asymptotes of $f(x) = \frac{x-1}{2x^2 - x - 1}$. In particular, evaluate $\lim_{x\to 1} f(x)$ to show that f *does not* have a vertical asymptote at $x = 1$.

4. Calculate $\lim_{\theta\to \pi^-} \cot\theta$ and $\lim_{\theta\to \pi^+} \cot\theta$. What can we say about the function $\cot\theta$?

5. (a) Find $\lim_{x\to 1} (3\csc 4(x-1) - 5)$

(b) More generally, if a, b, h, k are constants, find all vertical asymptotes of the function $y = a\csc b(x-h) + k$

6. If possible, evaluate the following limits.

(a) $\lim_{x\to \infty} \frac{e^{-x}}{e^x}$ (b) $\lim_{x\to -\infty} \frac{e^{-x}}{e^x}$ (c) $\lim_{x\to \infty} (5 - \ln(2x-1))$

(d) $\lim_{x\to \infty} \frac{x^{7/4} + 3x^2 - 10x^{5/2}}{7x + 13x^{8/3}}$ (e) $\lim_{x\to -\infty} \frac{1 - x + 12x^2}{5x^2 + 3 + 10x}$ (f) $\lim_{x\to \infty} \frac{-5x^3 + 3x - 9}{6x^2 + 19}$

7. Find all vertical and horizontal asymptotes of the function $h(x) = \frac{10x^2 - 29x - 21}{2x^2 - x - 15}$.

8. Consider the function $f(x) = \begin{cases} \cos \pi x, & x < -2 \\ 3^{-x-1}, & -2 \le x \le 0 \\ 2, & x > 0 \end{cases}$

Sketch a graph of $y = f(x)$ and find the following limits, if they exist.

(a) $\lim_{x\to -2^-} f(x)$ (b) $\lim_{x\to -2^+} f(x)$ (c) $\lim_{x\to -2} f(x)$

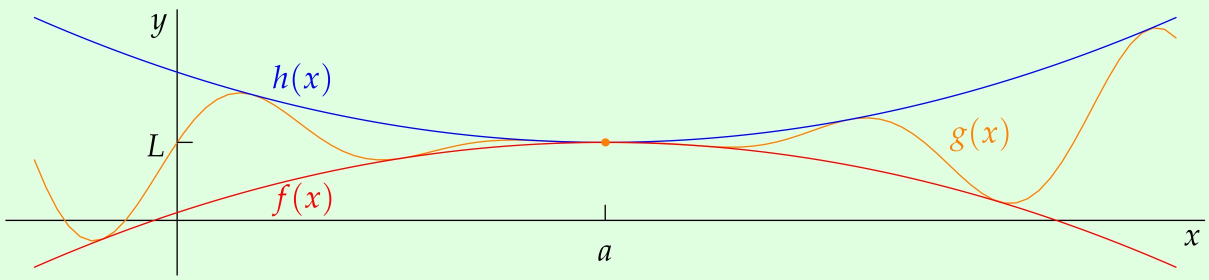
(d) $\lim_{x\to 0^-} f(x)$ (e) $\lim_{x\to 0^-} f(x)$ (f) $\lim_{x\to 0} f(x)$

1.2 Applications of Basic Limits and Continuity

The Squeeze Theorem

While simple limits may be computed using the basic limit laws, more complicated functions' limits may need to be calculated by *comparison*.

Theorem 1.12 (Squeeze Theorem). Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ in some open interval containing a, and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x)$ exists and is also equal to L.



The Squeeze Theorem tells us that a function g is forced to have the same limit as f and h if g is 'squeezed' between them, hence the name.

The Theorem also holds for limits at infinity.

Notice also the "for some *open interval containing a*". This open interval is typically all real numbers $(-\infty, \infty)$, but as we'll see in the following example, as long as a is within the interval, the conclusion holds.

Example 1.13. We use the Squeeze Theorem to calculate $\lim_{x\to\infty} \frac{\cos x}{x}$

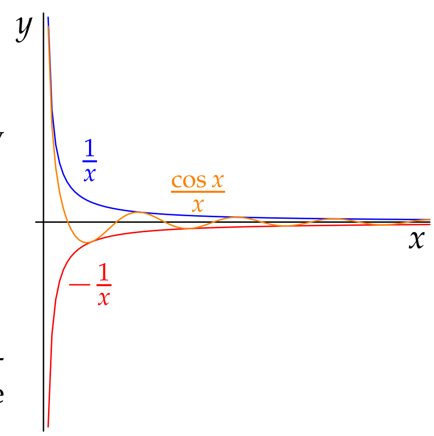
$-1 \leq \cos x \leq 1 \quad \text{for all } x > 0 \quad \implies -\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$

Since we are working with $x > 0$, we do not need to flip the inequality symbols when we divide by x.

Also $\lim_{x\to\infty} -\frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} = 0$

By the Squeeze Theorem, $\lim_{x\to\infty} \frac{\cos x}{x} = 0$

The choice of bounding functions $h(x)$ and $f(x)$ are somewhat flexible, so long as we know that $g(x)$ is between them. For instance, we could have chosen $\frac{2}{x}$ and $-\frac{2}{x}$ to compare with $\frac{\cos x}{x}$.



The Fundamental Trigonometric Limit

Consider $\lim_{x\to 0} \frac{\sin x}{x}$. We cannot simply substitute in $x = 0$ or do any simplification, and none of our limit laws can help us determine whether this limit exists or calculate it. We can use the Squeeze Theorem, however.

Theorem 1.14 (Fundamental Trigonometric Limit). $\lim_{x\to 0} \frac{\sin x}{x} = 1$

There are several ways to prove the Fundamental Trigonometric Limit with the Squeeze Theorem, but most of them require a geometric argument, which is not within the scope of this course. If you're curious, however, feel free to look up a proof (in particular, graph the functions $\cos x$, $\frac{\sin x}{x}$, $\sec x$ near $x = 0$).

We can combine our knowledge of the basic limit laws and the Fundamental Trigonometric Limit to evaluate some elaborate limits.

Examples 1.15. 1. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = \lim\limits_{x\to 0} \dfrac{3}{3}\dfrac{\sin 3x}{x} = 3 \lim\limits_{x\to 0} \dfrac{\sin 3x}{3x} = 3(1) = 3$

Notice that when the *argument* of sin is equal to the denominator, the limit will be 1.

2. In general, for a and b constant,

$$\lim_{x\to 0} \frac{\sin ax}{bx} = \lim_{x\to 0} \frac{a/b}{a/b}\frac{\sin ax}{bx} = \frac{a}{b}\lim_{x\to 0} \frac{\sin ax}{ax} = \frac{a}{b}(1) = \frac{a}{b}$$

3. $\lim\limits_{x\to 0} \dfrac{x}{\sin x} = \lim\limits_{x\to 0} \dfrac{1}{\frac{\sin x}{x}} = \lim\limits_{x\to 0} 1 \div \lim\limits_{x\to 0} \dfrac{\sin x}{x} = \dfrac{1}{1} = 1$

4. $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{5x^2} = \lim\limits_{x\to 0} \dfrac{(\sin 3x)(\sin 3x)}{5(x)(x)} = \lim\limits_{x\to 0} \dfrac{\sin 3x}{5x} \cdot \lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = \dfrac{3}{5}(3) = \dfrac{9}{5}$

Continuity

A crucial concept in calculus is *continuity* of functions. Many theorems and results have continuity as a condition.

Definition 1.16 (Continuity). A function f is *continuous* at $x = c$ if:

1. $f(c)$ is defined
2. $\lim\limits_{x\to c} f(x)$ exists
3. $\lim\limits_{x\to c} f(x) = f(c)$

If f is continuous for all x-values on an interval $[a, b]$, we say f is continuous on $[a, b]$. Similarly, if f is continuous for all real numbers $(-\infty, \infty)$, we simply say that f is continuous.

If any of the three conditions fails, we say that f is *discontinuous* at $x = c$. Alternatively, f has a *discontinuity* or *is not continuous* at $x = c$.

The informal idea of a continuous function is that one can draw the graph of the function without taking one's pen off the paper. Remember this; it is very useful to visualize this idea in future definitions and theorems requiring continuity!

However, never use this as justification in questions which relate to continuity! If you want to show that a function is continuous or discontinuous at $x = c$, use Definition 1.16.

Examples 1.17. 1. Determine whether the function f defined by

$$f(x) = \begin{cases} -\frac{1}{4}x^3 + 1, & x < 0 \\ 2x^{1/3} + 1, & x \geq 0 \end{cases}$$

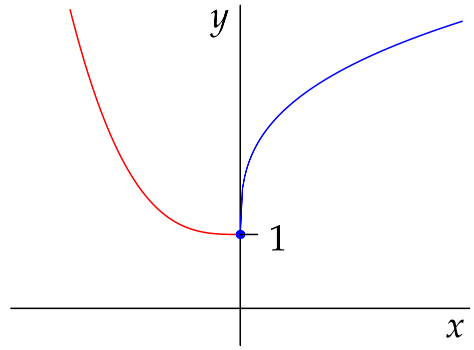
is continuous at $x = 0$. We check using the definition:

$f(0)$ certainly exists.

$\lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x) \implies \lim_{x \to 0} f(x)$ exists.

$f(0) = \lim_{x \to 0} f(x) = 1$

Thus f is continuous at $x = 0$.



2. Check $g(x) = \begin{cases} x^2 - 3, & x \neq -1 \\ 2, & x = -1 \end{cases}$ for continuity at $x = -1$.

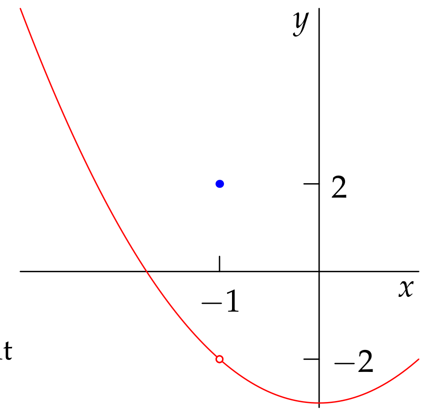
   $g(-1)$ is defined.

   $\lim_{x \to -1^-} g(x) = 0 = \lim_{x \to -1^+} g(x) \implies \lim_{x \to -1} g(x)$ exists.

   $g(-1) = 2 \neq \lim_{x \to -1} g(x) = -2$

   The third condition fails, so g is discontinuous at $x = -1$.

   Notice that we cannot draw the graph of g at $x = -1$ without lifting our pen.

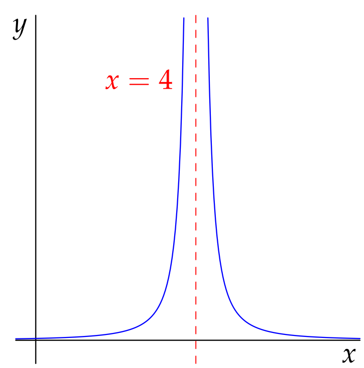


3. Is $h(x) = \dfrac{1}{x^2 - 8x + 16}$ continuous at $x = 4$?

   $h(4)$ is not defined.

   Since the first condition fails, we do not need to continue with the work; we are done! h is not continuous at $x = 4$.

   Visually, h has a vertical asymptote at $x = 4$. Of course, we cannot draw the graph of h at $x = 4$ without lifting up our pen; the left and right branches do not connect.



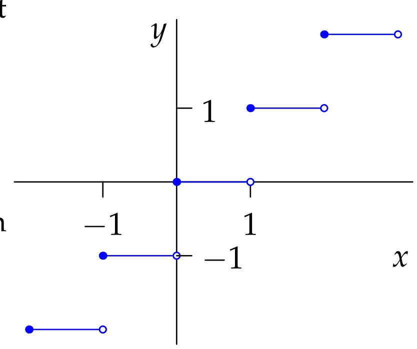
4. The greatest integer function $r(x) = \lfloor x \rfloor$ has discontinuities at each integer. We can check, for example, at $x = 2$:

   $r(2)$ is defined.

   $\lim_{x \to 2^-} r(x) = 1 \neq \lim_{x \to 2^+} r(x) = 2 \implies \lim_{x \to -1} r(x) = \text{DNE}$

   The limit of r as $x \to 2$ does not exist, so the second condition fails. Therefore r is discontinuous at $x = 2$.

   We could use a similar argument for other integers as well.



Types of Discontinuities

As the investigation in Examples 1.17 demonstrate, there are several types of discontinuities we may study in AP Calculus.

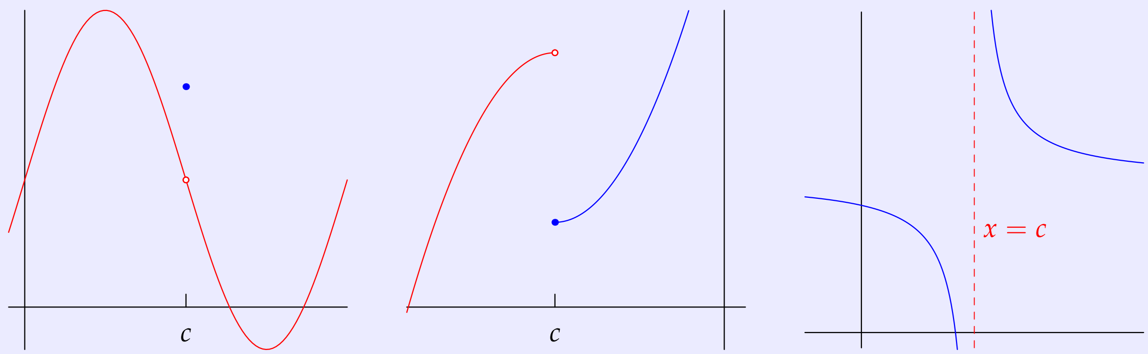
Definition 1.18. Suppose the function f has a discontinuity at $x = c$. We may classify it as such:

If $\lim_{x\to c} f(x)$ exists $\iff \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$, we call it a *removable discontinuity*.

If $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist but $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$, we call it a *jump discontinuity*.

If $\lim_{x\to c^\pm} f(x) = \pm\infty$ (i.e. f has a vertical asymptote at $x = c$), we call it an *infinite discontinuity*.[5]

The latter two are *non-removable discontinuities*.



Removable discontinuity | Jump discontinuity | Infinite discontinuity

The removable discontinuity is appropriately named; if we simply "fill in" the hole, then we could draw the graph of the function at $x = c$ without lifting our pen. That is, the function would become continuous.

Theorem 1.19. Suppose f has a removable discontinuity at $x = c$, and $\lim_{x\to c} = L$. Then

$\tilde{f}(x) = \begin{cases} f(x), & x \neq c \\ L, & x = c \end{cases}$ is continuous at $x = c$.

We want to set $\tilde{f}(c)$ to the limit of f at $x = c$ so that we fulfill the third (and perhaps first) condition for continuity.

We can also return to Examples 1.17 to classify each discontinuity.

The function g has a removable discontinuity at $x = -1$. If we simply let $\tilde{g}(x) = x^2 - 3$, then $\tilde{g}$ will be continuous on $(-\infty, \infty)$.

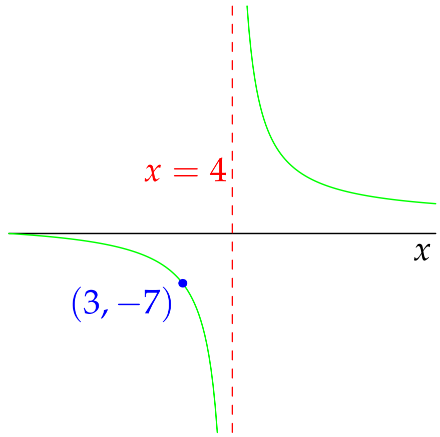
The function h has an infinite discontinuity at $x = 4$ since it has a vertical asymptote at that x-value.

The floor function $\lfloor x \rfloor$ has a jump discontinuity at every integer value of x.

[5]Some authors will also call it an *essential discontinuity*.

Example 1.20. Suppose that $f(x) = \dfrac{2x^2 - 5x - 3}{x^2 - 7x + 12}$.

Re-assign a value for f at any removable points of discontinuity so that it is continuous at that x-value.



First, $f(x) = \dfrac{2x^2 - 5x - 3}{x^2 - 7x + 12} = \dfrac{\cancel{(x-3)}\,(2x+1)}{\cancel{(x-3)}\,(x-4)} = \dfrac{2x+1}{x-4}, \quad x \neq 3$

$\implies \lim\limits_{x \to 3} f(x) = \dfrac{2\,(3)+1}{3-4} = -7$

$\implies \tilde{f}(x) = \begin{cases} \frac{2x^2-5x-3}{x^2-7x+12}, & x \neq 3 \\ -7, & x = 3 \end{cases}$ is continuous at $x = 3$.

f also has an infinite discontinuity at $x = 4$, but it is non-removable.

There are many continuous functions! If continuity is sufficient for the conclusion of a theorem, of course it is impossible to show that a function is continuous at each and every x-value using the three-part definition. We instead use the following theorem.

Theorem 1.21. The following functions are continuous everywhere they are defined (i.e. everywhere in their domain):

1. Polynomials
2. Power functions
3. Rational functions
4. Exponential functions
5. Logarithmic functions
6. Trigonometric functions
7. Sums, differences, products, quotients, and compositions of continuous functions

Be wary of the "everywhere they are defined"! For example, $\ln x$ is not continuous at $x = 0$. Indeed, it is not defined on the interval $(-\infty, 0]$!

In essence, any 'nice' combination of the elementary functions you have studied in prerequisite courses are continuous if you can find their domain.

Examples 1.22. 1. $f(x) = \cos\left(\sqrt{x^2+1}\right)$ is continuous everywhere.

2. $g(x) = 5\ln(3x-1) + 2$ is defined when $3x - 1 > 0 \implies x > \frac{1}{3}$, so it is continuous on the interval $(\frac{1}{3}, \infty)$.

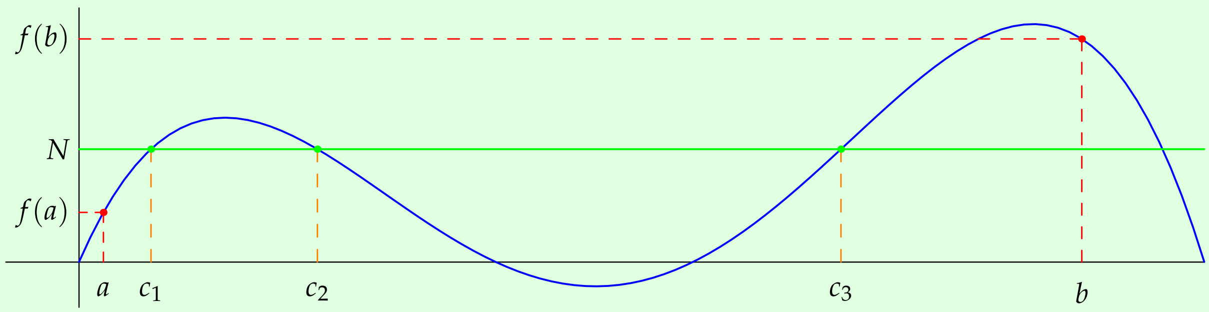
3. $h(x) = (1 - \sin x)^{-3}$ is continuous except when $\sin x = 1$, i.e. when $x = \frac{\pi}{2} + 2\pi n$ for any integer n.

If you are finding it difficult to find the domain of these functions, be sure to brush up on that content!

The Intermediate Value Theorem

This is our first important result which uses continuity as a condition.

Theorem 1.23 (Intermediate Value Theorem). Suppose f is continuous on $[a, b]$, that $f(a) \neq f(b)$, and that N lies between $f(a)$ and $f(b)$. Then there is some value c between a and b such that $f(c) = N$.



Roughly speaking, the Intermediate Value Theorem (IVT) tells us if we can draw the graph of a function without lifting up our pen (the function is continuous), then the graph must have passed through every y-value between the endpoints of our interval. Do your best to visualize this.

As the picture shows, there may be more than one choice of c, but IVT only guarantees at least one.

Also notice that in the picture, there seems to be another x-value outside the interval $[a, b]$ that intersects the line $y = N$. However, we don't care, and we aren't even sure it exists! IVT is only concerned with values within the given interval.

There are many ways to interpret and apply IVT, for example:

- If the temperature at 6 a.m. is 65° F and the temperature at 12 p.m. is 95° F, then at some time during the morning, the temperature must have been 80° F.
- If you climb a 10,000 ft mountain starting from sea level, then at some point, you must be at an elevation of exactly 7,540 ft.

In mathematics, the Theorem is often used to show that certain equations have solutions, and to then use algebra or otherwise find those solutions.

Example 1.24. The table below shows selected values of a polynomial function p. For $-2 \leq x \leq 12$, what is the fewest possible number of zeros of p?

x	-2	0	1	3	4	5	8	9	11	12
$p(x)$	5	-2	1	7	13	35	14	-5	-2	6

Since p is a polynomial, it is continuous for all real numbers. Thus, the conclusion of IVT holds. In particular, since $f(-2) < 0$ and $f(0) > 0$, for *at least one* x-value c_1 between $x = -2$ and $x = 0$, $f(c_1) = 0$. A similar argument can be made to show that there exists at least one zero between $x = 0$ and 1, 8 and 9, as well as 11 and 12. So the fewest possible number of zeros of p on $[-2, 12]$ is *four*.

Exercises 1.2. 1. Test explicitly for the continuity of $f(x) = \sqrt{9-x^2}$ at $x = -2$ using the three-part definition.

2. (a) State the *range* of the function $\cos\left(\frac{1}{x^2}\right)$.

   (b) Hence find $\lim\limits_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$.

   (*Hint: x^2 is always positive; think about what this does to the inequality*)

3. Evaluate the following limits if possible:

   (a) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{3x}$ (b) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$ (c) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{9x}$

   (d) $\lim\limits_{x\to 0} x \cot x$ (e) $\lim\limits_{x\to 0} \dfrac{x^2+x}{\sin 2x}$ (f) $\lim\limits_{x\to 0} 2x \csc 3x$

4. Calculuate $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$ by multiplying by $\dfrac{1+\cos x}{1+\cos x}$.

5. Identify *all* discontinuities of $h(x) = \dfrac{x^3-125}{2x^2-3x-35}$, and re-assign function values at any *removable* discontinuities.

6. Find the values of a and b for which the following function g is continuous everywhere:

$$g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x < 1 \\ ax+b, & 1 \le x < 3 \\ 3^x - 10, & x \ge 3 \end{cases}$$

7. Determine the intervals for which $f(x) = \cos x + 3x^2 + 17(x+4)^{5/2} - \dfrac{6x^{8/3}-9}{x-3}$ is continuous.

8. The water level of a city's reservoir is modeled by the continuous function W, where $W(t)$ is measured in meters and t is measured in number of hours since noon ($t = 0$). Selected values of $W(t)$ are shown in the table below.

t (hours)	0	1.3	2.7	4.4	5.9	7.5	9
$W(t)$ (meters)	50	83	125	71	52	89	130

   Between noon and 9 p.m., what is the fewest number of times the depth of the reservoir was exactly 75 m? Justify your answer.

9. (Calculator) Consider the function $f(x) = x^3 + \tan x + 1$.

   (a) Find $f(-1)$ and $f(0)$.

   (b) Show that the equation $f(x) = 0$ has at least one solution on the interval $(-1, 0)$.

   (c) Use your calculator to approximate this solution to 3 decimal places.

10. $g(x) = x + \frac{1}{x}$ satisfies $f(-1) < 0$ and $f(1) > 0$. Can we conclude that there is some value c on the interval $(-1, 1)$ for which $f(c) = 0$? Why or why not?

11. (Hard) Suppose that functions g and h are continuous with $g(3) = h(3) = 5$.

    (a) It is known that $g(x) \leq h(x)$ for $2 < x < 4$. Let k be a function satisfying the inequality $g(x) \leq k(x) \leq h(x)$ for $2 < x < 4$. Is k continuous at $x = 3$? Justify your answer.

    (b) Let f be the function given by $f(x) = 5x^2 h(x) - \dfrac{1}{25 - (k(x))^2}$. Is f continuous at $x = 3$? Explain your reasoning.

2 Differentiation

2.1 The Derivative of a Function

Students should hopefully be somewhat familiar with the idea of *rate of change*. The concept is often introduced with *slope* of a line, often notated m. We should be used to the formula

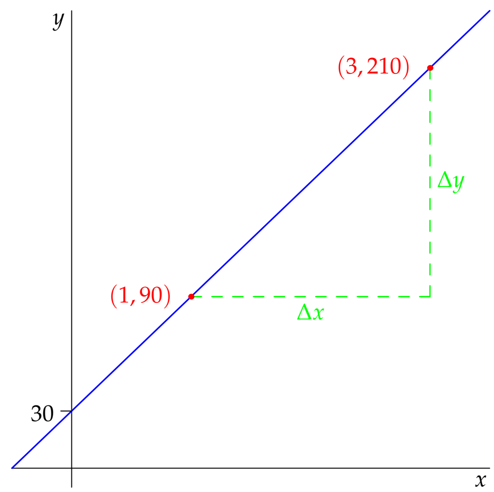
$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Think deeply about what exactly this formula is computing. $\frac{\text{rise}}{\text{run}}$ is the *ratio* of how quickly the y-value is changing in comparison to the x-value. Δy and Δx are the 'change in y' and 'change in x' respectively. By dividing these two quantities, we obtain the rate at which y changes when x increases by exactly one unit. It may help to think in terms of an example.

Example 2.1. Suppose I am driving on a straight path away from my house at a constant speed. After 1 hour of driving, I am 90 miles from my house. 3 hours after beginning my drive, I am 210 miles from my house. If we view "time in hours" as x and "distance from my house in miles" as y, then we can use the *coordinates* $(1, 90)$ and $(3, 210)$ to compute slope:

$$m = \frac{210 - 90}{3 - 1} = 60 \text{ miles/hour}.$$

Notice the *unit* miles/hour: it is indeed a *rate*. What we are saying is that, for every 1 hour that passes, I travel an additional 60 miles away from my house.
We can further find the y-intercept $b = 30$, which yields the linear equation $y = 60x + 30$.



This may seem highly trivial, but things get tricky when we begin to think about rates of change for *non-linear* functions. Notice that we can choose *any* two points on a line to substitute into our slope formula; if we had chosen $(0, 30)$ and $(5.5, 360)$ for instance, we would have ended up with the same value for m. That is, the slope of a line is *constant*. We consider an example of a non-linear function.

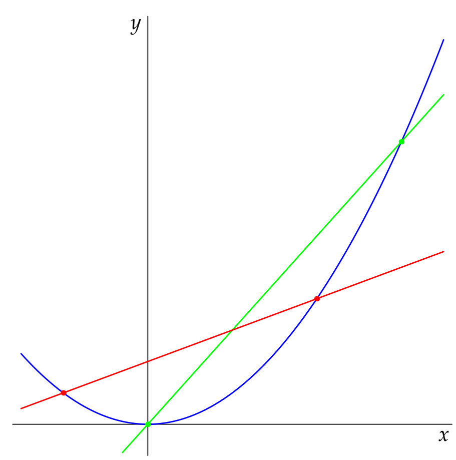
Example 2.2. Consider $f(x) = x^2$.
We first choose the points $(0, 0)$ and $(3, 9)$ on the parabola to calculate slope:

$$m = \frac{f(3) - f(0)}{3 - 0} = \frac{9 - 0}{3 - 0} = 3$$

By contrast, we can choose $(-1, 1)$ and $(2, 4)$:

$$m = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - 1}{2 - (-1)} = 1$$

Seemingly, the slope of x^2 *changes* as we move along the curve.

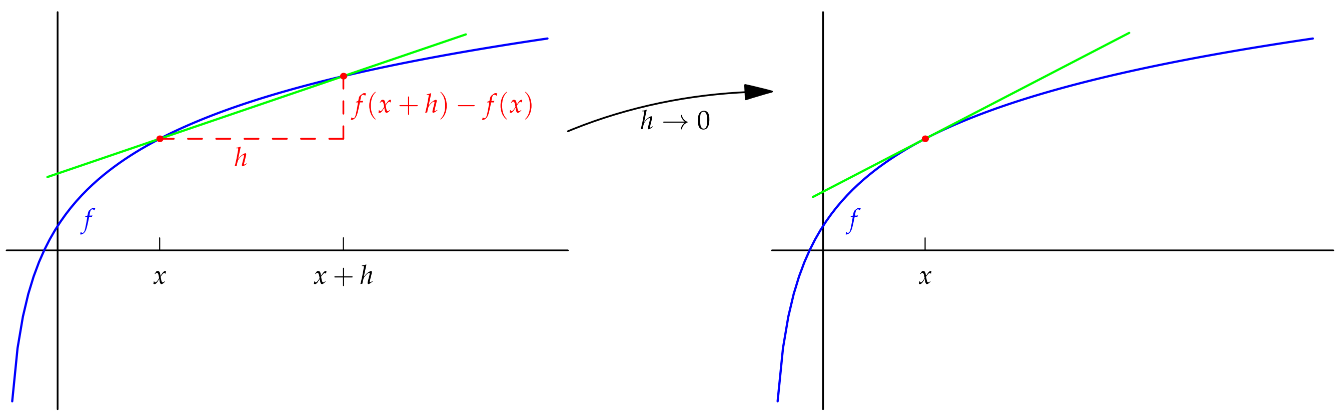


What we did in Example 2.2 was calculate *average slope* or *average rate of change* over an interval. In essence, we choose two points $(a, f(a))$ and $(b, f(b))$ to construct a *secant* (cutting) line connecting the two points and measure its slope to describe the average rate of change of a function $\frac{f(b)-f(a)}{b-a}$ over an *interval* $[a, b]$.

For a function f, we can choose any point $(x, f(x))$ on the graph of f and point $(x+h, f(x+h))$ which is h units away horizontally from x. The average slope between those two points is then:

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}.$$

What then happens if we bring our two chosen points closer together? More specifically, *infinitely close* together? That is, taking the *limit* of slope expression as $h \to 0$; we are closing in on the definition of *instantaneous* slope. Instead of constructing a secant line to measure slope over an interval, we are constructing a *tangent* (touching) line to measure slope at a *single point* on the graph of f.



Definition 2.3 (Limit Definition of the Derivative). The *instantaneous rate of change* function of f is the slope of the tangent line to the graph $y = f(x)$ at any given x-value, if such a tangent line exists. It is termed the *derivative of* f, notated f', and is given by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Alternatively, if we are given y in terms of x it can be denoted $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ or y'.

The process of finding derivative functions is called *differentiation*[6]. In particular, $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x)$ is *differentiation with respect to* x; $f'(x)$ is the *derivative with respect to* x of $f(x)$.

The value of the derivative evaluated at $x = c$ is

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

Other common notations for the derivative at $x = c$ are $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=c}$ and $y'(c)$.

[6]Beware, although completely intuitive, the process of finding derivatives is *not* called 'deriving'!

Notation

The two mathematicians most credited with the development of Calculus, Issac Newton and Gottfried Wilhelm Leibniz, had different notations for derivative. The f' notation is a modification of Newton's style, while $\frac{\mathrm{d}y}{\mathrm{d}x}$ is Leibniz's notation. Be familiar with both; each has their own pros and cons. In particular, Leibniz's approach importantly reminds us what derivatives are: rates of change of one quantity with respect to another. This is absolutely crucial: never forget that derivatives represent *instantaneous rate of change*!

The language 'with respect to $\square$' is also quite important. What it means is that if we differentiate a function with respect to $\square$, then the value of that function changes by some amount as $\square$ changes by one unit.

In practice, finding derivatives is quite easy when we can use special rules which will be introduced later, but for now, we will need to use a lot of algebra, ingenuity, and our limit laws, since we cannot just substitute in $h = 0$ immediately (divide by 0!).

Examples 2.4. 1. Differentiate $f(x) = 3x^2$ with respect to x.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3(x^2+2xh+h^2)-3x^2}{h} \\ &= \lim_{h\to 0} \frac{3x^2+6xh+3h^2-3x^2}{h} = \lim_{h\to 0} \frac{6xh+3h^2}{h} = \lim_{h\to 0} \frac{h(6x+3h)}{h} \\ &= \lim_{h\to 0}(6x+3h) = 6x+3(0) = 6x \end{aligned}$$

2. If $y = \frac{1}{x}$, find $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3}$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{x^2+xh}\cdot\frac{1}{h} \\ &= \lim_{h\to 0} \frac{-1}{x^2+xh} = -\frac{1}{x^2+x(0)} = -\frac{1}{x^2} \\ &\implies \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=3} = -\frac{1}{3^2} = -\frac{1}{9} \end{aligned}$$

As you can see, finding derivatives in this manner is very tedious and requires considerable knowledge of algebraic manipulation. Once again, if you are not comfortable with these examples, touch up on those subjects. Namely, trigonometric identities, combining fractions, expanding polynomials, etc.

As Examples 2.4 should demonstrate, the language for derivatives might take some getting used to. The symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ can be thought of as an *operator*: it takes a function and turns it into its derivative.

Examples 2.4 cont. If $y = f(x) = 3x^2$, then we know $f'(x) = 6x$. We could also write $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x$, or $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2\right) = 6x$.

Moreover $f'(-2) = 6(-2) = -12$ and $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=4} = 6(4) = 24$.

Otherwise said, the slope of the tangent line to $f(x) = 3x^2$ at $x = -2$ is -12, and the slope of the tangent line to $y = 3x^2$ at $x = 4$ is 24.

Equation of the Tangent Line

Sometimes, we may be asked to find the *equation* of the tangent line to a curve at a particular point. Recall the *point-slope form* of the equation of a line:

$$y - y_0 = m(x - x_0)$$

where (x_0, y_0) is a *given point* and m is the *slope* of the line. Essentially, our goal is to find those three constants x_0, y_0, and m.

We will often be asked to construct a tangent line to the graph of $y = f(x)$ at the $x = a$. Then those three constants we find will change in context:

$(x_0, y_0) = (a, f(a))$; $x = a$ will always be given, so simply substitute a into $f(x)$ to find $y_0 = f(a)$.

m is the slope of line we construct, which is the tangent line. Observant students will see that the slope of the tangent line is the *derivative* of $y = f(x)$ at $x = a$. That is, $m = f'(a) = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a}$.

Theorem 2.5. The tangent line to the graph of $y = f(x)$ at $x = a$ has equation $y - f(a) = f'(a)(x - a)$.

We may also be asked to find the equation of the *normal line*, which is the line *perpendicular* to the tangent line passing through the given point. Its (x_0, y_0) are the same, but its slope is the negative reciprocal of the slope of the tangent. That is, $-\frac{1}{m}$.

Example 2.6. Find the equation of the tangent line and normal line to $f(x) = 2x^2$ at $x = -3$.

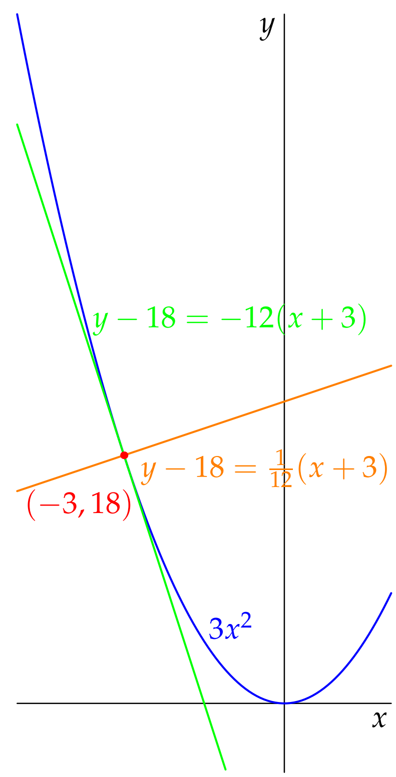
We start with the point-slope equation $y - y_0 = m(x - x_0)$.
We are given $x = -3$, so $x_0 = -3$.
$\implies y_0 = f(x_0) = f(-3) = 2(-3)^2 = 18$.
To find m and $-\frac{1}{m}$, we need the derivative f':

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{2(x+h)^2 - 2x^2}{h} = \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) - 2x^2}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2}{h} = \lim_{h\to 0} \frac{h(4x + 2h)}{h} \\ &= \lim_{h\to 0} (4x + 2h) = 4x + 2(0) = 4x \\ &\implies m = f'(-3) = 4(-3) = -12 \\ &\implies -\frac{1}{m} = \frac{1}{12} \end{aligned}$$

Now that we have the required constants, we can substitute into our formulas:

Tangent: $y - 18 = -12(x + 3)$ Normal: $y - 18 = \frac{1}{12}(x + 3)$

And we are done! No need to simplify further (unless asked).



Notice that in this picture, the tangent line is merely *touching* the graph of $y = f(x)$ at the point $(-3, 18)$, and its slope seems to match the rate of change of the graph. The normal line also passes through the given point, but is *perpendicular* to the tangent.

Differentiability

Unfortunately, not every function's derivative is defined at all points.

Definition 2.7. A function f is *differentiable* at $x = c$ if its derivative is defined at $x = c$. In particular, the following limit exists and is finite:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

If f is differentiable for all x-values on an interval $[a, b]$, we say f is differentiable on $[a, b]$. Similarly, if f is differentiable for all real numbers $(-\infty, \infty)$, we simply say that f is differentiable.

If the above limit does not exist or is infinite, we say that f is *non-differentiable* at $x = c$.

Remember that the derivative is defined as a *limit*; it exists if and only if the left- and right-side limits are equal. This leads to the following result.

Theorem 2.8. A function f is differentiable at $x = c$ if and only if

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \iff \lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$$

We can test if a function is differentiable at any point using all of these limits, but what does a differentiable function or non-differentiable function look like? We want to develop an intuitive sense[7] for what we should expect the graph of a differentiable function to look like. So we should start by considering the ways in which a function might *fail* to be differentiable. The simplest approach is related to continuity.

Theorem 2.9. If f is differentiable at $x = c$, then f is continuous at $x = c$. Equivalently, if f is discontinuous at $x = c$, then f is non-differentiable at $x = c$.

So, as it turns out, the simplest way to check for non-differentiability is to first check for discontinuity.

Example 2.10. Verify that $g(x) = 9\tan\left(\frac{1}{2}x - \pi\right)$ is non-differentiable at $x = 3\pi$.

We first check to see if g is discontinuous at $x = 3\pi$ using the three-part definition:

$g(3\pi) = 9\tan\left(\frac{1}{2}(3\pi) - \pi\right) = 9\tan\frac{\pi}{2}$ is undefined since $\cos\frac{\pi}{2}$ is 0.

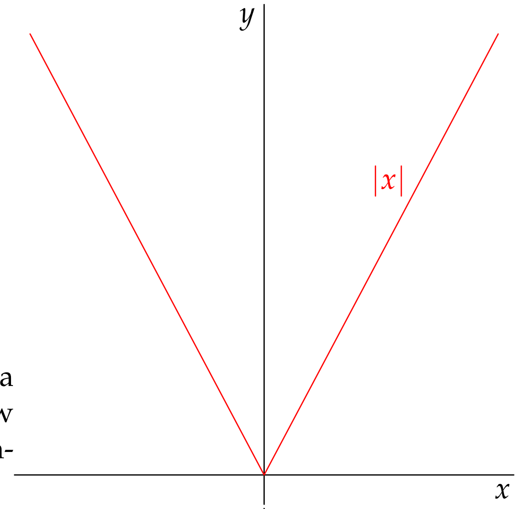
The first condition for continuity of g fails at $x = 3\pi$, so it is discontinuous at $x = 3\pi$. Therefore it is also non-differentiable at $x = 3\pi$.

In fact, g has a vertical asymptote (and therefore an infinite discontinuity) at $x = 3\pi$.

Every differentiable function is necessarily continuous, but the converse is false. It is possible for a function to be continuous but not differentiable. For this to happen, the limit in Definition 2.7 cannot exist. There are several common possibilities.

[7]Similarly to how a continuous function should be able to be drawn without lifting your pen from the page.

Examples 2.11. *Corner* Consider $f(x) = |x|$. We can write it as



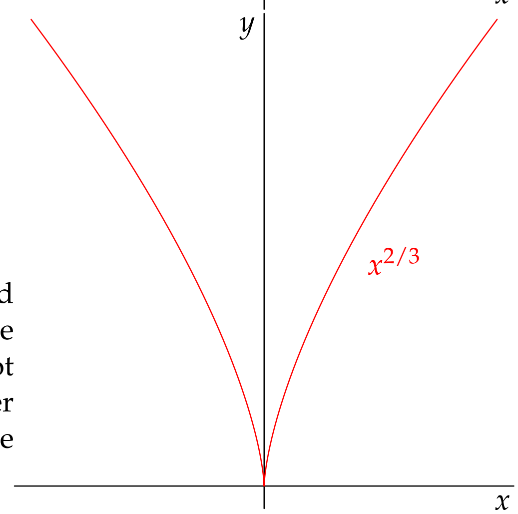
$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

If we differentiated both sides (check this!), we would get

$$f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases} \implies \lim_{x \to 0^-} f'(x) \neq \lim_{x \to 0^+} f'(x)$$

In essence, the curve enters the point $(0,0)$ and leaves in a different direction; we call it a *corner*. Where should we draw a tangent line if one existed? Indeed, we cannot. $|x|$ is continuous but *not differentiable* at $x = 0$.

Cusp We check if $g(x) = x^{2/3}$ is differentiable at $x = 0$:

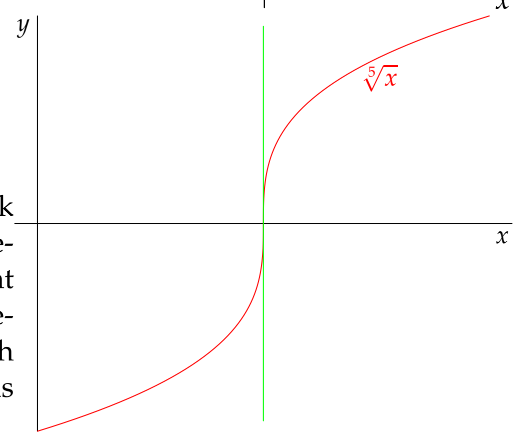


$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^{2/3}}{h} = -\infty,$$ [8]

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^{2/3}}{h} = \infty$$

As per the definition, the one-sided limits are infinite, and the overall limit does not exist. So $x^{2/3}$ is not differentiable at $x = 0$. Once again, the function is continuous but not differentiable at a point. The origin is a special type of corner called a *cusp*, where the curve leaves the point in the opposite direction to how it entered.

Vertical Tangent Test $y = \sqrt[5]{x-1}$ for differentiability at $x = 1$.



$$\lim_{h \to 0} \frac{y(1+h) - y(1)}{h} = \lim_{h \to 0} \frac{\sqrt[5]{h}}{h} = \lim_{h \to 0} h^{-4/5} = \infty$$

The limit is infinite. What does this mean? If we take a look at the graph, the *tangent line* at $x = 1$ seems to be *vertical*. Remember that derivatives are defined as slope of the tangent line; and a vertical line has a slope which is *undefined*. Therefore, the derivative is also not defined at any value for which the tangent line is vertical. So, in this example, $y = \sqrt[5]{x-1}$ is continuous at $x = 1$ but not differentiable.

What should we take from these examples? Perhaps a good intuitive guide for what the graph of a differentiable function looks like is that it is *smooth*[9]: it contains no sharp turns and has no vertical tangents.

[8] You can graph the function $\frac{h^{2/3}}{h}$ if you want to be convinced of these limits; same goes for the limits below.

[9] The meaning of *smooth* depends on the author. Elsewhere, we might say that a smooth function is *infinitely differentiable*.

Higher Order Derivatives

We can differentiate derivatives! The process is the same as previously.

Definition 2.12. The *second derivative* of f is the derivative of f': that is

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

if the limit exists. In Leibniz notation, we write as if we are *squaring the derivative operator*:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y$$

We can similarly evaluate higher order derivatives:

$$\begin{aligned} &\text{Third:} && f'''(x) = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} \\ &\text{Fourth:} && f^{(4)}(x) = \frac{\mathrm{d}^4 y}{\mathrm{d}x^4} \\ &\text{Fifth:} && f^{(5)}(x) = \frac{\mathrm{d}^5 y}{\mathrm{d}x^5} \end{aligned}$$

and so on. If the second derivative of f exists, we say f is *twice-differentiable*. If the n^{th} order derivative of f exists, we say f is *n-times-differentiable*. If every order derivative exists for f, then we say that f is *infinitely differentiable*.

Example 2.13. If $f(x) = 5x^2 - 2x$, find $f'''(x)$ using the limit definition of the derivative.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\left[5(x+h)^2 - 2(x+h)\right] - \left[5x^2 - 2x\right]}{h} = \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 2x - 2h - 5x^2 + 2x}{h} \\ &= \lim_{h \to 0} \frac{h\,(10x + 5h - 2)}{h} = \lim_{h \to 0} (10x + 5h - 2) = 10x - 2 \\ f''(x) &= \lim_{h \to 0} \frac{[10(x+h) - 2] - [10x - 2]}{h} = \lim_{h \to 0} \frac{10x + 10h - 2 - 10x + 2}{h} \\ &= \lim_{h \to 0} \frac{10h}{h} = \lim_{h \to 0} 10 = 10 \\ f'''(x) &= \lim_{h \to 0} \frac{10 - 10}{h} = \lim_{h \to 0} \frac{0}{h} = 0 \end{aligned}$$

Hopefully you are beginning to re-develop algebra skills. And perhaps you may be starting to spot a pattern with some of these derivatives. . .

Exercises 2.1. 1. Explain why $y = 1 - |x+2|$ is not differentiable at $x = -2$. Is it continuous at $x = -2$?

2. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following functions:

   (a) $y = 15$ (b) $y = 6x$ (c) $y = 7 - 2x$

3. For each function in the previous problem, find $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1}$, if it exists.

4. Verify that $g(x) = \dfrac{x-6}{x^2-36}$ is not differentiable at $x = 6$ and $x = -6$.

   (*Hint: Do not try to find $g'(x)$!*)

5. Find the slope of the tangent line to $f(x) = -4x^2$ at $x = 1$.

6. Find the equation of the tangent line to $y = -\frac{3}{x}$ at $x = -2$ and sketch a graph of both the function and the tangent line.

7. (a) Find $\displaystyle\lim_{h\to 0} \frac{\cos h - 1}{h}$ by multiplying by $\dfrac{\cos h + 1}{\cos h + 1}$

   (b) Hence find the derivative of $f(x) = \sin x$.

   (c) If you're ambitious, try to find the derivative of $g(x) = \cos x$ as well!

8. If $y = x^3$, find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

2.2 Basic Rules for Differentiation

Hopefully you agree after the previous section that finding derivatives manually with the limit definition is extremely tiresome. Thankfully, we will develop some shortcuts to differentiate certain functions. There are many derivative rules, but do your best to memorize them! It takes plenty of time and practice to do so. Being able to quickly and efficiently differentiate functions is a key skill for the AP exam.

The first rule is simple.

Theorem 2.14. If $f(x) = c$ is constant, then its derivative is

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}c = 0$$

Proof. Here are a few possible arguments.

1. If c is constant, then the graph of $f(x) = c$ is a straight horizontal line. Its slope at any point (i.e. its derivative) must be 0.

2. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$

Another quick proof is sketched in an exercise. ■

Example 2.15. The derivative of $y = -2$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

The Power Rule

You may have noticed from the previous section that derivatives of *polynomials* follow a certain pattern. It can be generalized as follows.

Theorem 2.16 (Power Rule). Let $f(x) = x^n$, where n is constant. Then its derivative is

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

In short, if we have x to a constant power, when we differentiate, we can bring that power down to the coefficient and subtract one from it. Notice also that we did not specify that f is a polynomial; the power rule works for x to *any* constant power, including negatives and fractions!

Examples 2.17. 1. If $f(x) = x^{17}$, then $f'(x) = 17x^{17-1} = 17x^{16}$.

2. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

3. If $y = x^{4.9513}$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 4.9513x^{3.9513}$

4. The derivative of $g(x) = x^{\pi}$ is $g'(x) = \pi x^{\pi-1}$

5. We find the slope of the tangent line to $y = \frac{1}{x^2}$ at $x = -2$.

$$y = \frac{1}{x^2} = x^{-2} \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-3} = -\frac{2}{x^3} \Longrightarrow \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-2} = \frac{2}{(-2)^3} = -\frac{1}{4}$$

The primary takeaway here is that if you encounter power terms in the denominator or roots, converting them into a negative exponent or fraction exponent allows us to differentiate using the power rule.

Linearity

Theorem 2.18 (Linearity). If f, g are functions and a, b are constants, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(af(x) + bg(x)) = af'(x) + bg'(x)$$

This Theorem simply says that constant coefficients may be ignored in differentiation, and that sums and differences of functions can be differentiated separately.

Linearity together with the power rule allows us to differentiate any polynomial.

Examples 2.19. 1. $\frac{\mathrm{d}}{\mathrm{d}x}(6x^2 - 7x^5 + 8) = 6(2)x^1 - 7(5)x^4 + 0 = 12x - 35x^4$

2. Find the equation of the tangent line to $y = 3\sqrt{x} - \frac{2}{x}$ at $x = 4$.

   The equation of the tangent line is $y - y_0 = m(x - x_0)$.

   We are given $x_0 = 4 \Longrightarrow y_0 = 3\sqrt{4} - \frac{2}{4} = 3(2) - \frac{1}{2} = \frac{11}{2}$, so $(x_0, y_0) = \left(4, \frac{11}{2}\right)$.

   Next, we convert $y = 3x^{1/2} - 2x^{-1}$. Then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^{-1/2} + 2x^{-2} = \frac{3}{2\sqrt{x}} + \frac{2}{x^2}$

   $$\Longrightarrow m = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=4} = \frac{3}{2\sqrt{4}} + \frac{2}{4^2} = \frac{3}{4} + \frac{2}{16} = \frac{7}{8}$$

   So the equation of the tangent line at $x = 4$ is $y - \frac{11}{2} = \frac{7}{8}(x - 4)$.

Derivatives of Elementary Functions

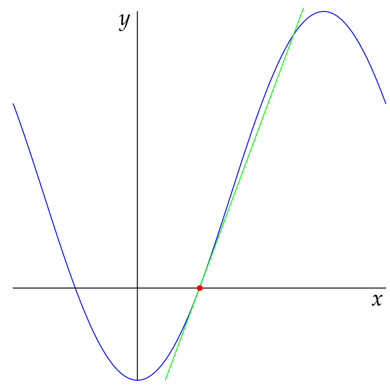
As we saw in the previous set of Exercises, $\sin x$ and $\cos x$ are differentiable and each have their own derivatives.

Theorem 2.20. Let $f(x) = \sin x$ and $g(x) = \cos x$. Then their derivatives are

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \quad \text{and} \quad g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

As said, the proofs are in the previous Exercises and require use of the Fundamental Trigonometric Limit.

Example 2.21. Find the equation of the tangent line to $f(x) = -2\cos x + 1$ at $x = \frac{\pi}{3}$.



First, $f(\frac{\pi}{3}) = -2\cos\frac{\pi}{3} + 1 = -2(\frac{1}{2}) + 1 = 0$.
Now $f'(x) = -2(-\sin x) + 0 = 2\sin x$
$\implies f'(\frac{\pi}{3}) = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$
So the equation of the tangent line to $y = f(x)$ at $x = \frac{\pi}{3}$ is:

$$y - 0 = \sqrt{3}(x - \tfrac{\pi}{3})$$

Theorem 2.22. If $f(x) = e^x$, then its derivative is:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$

The beauty of this theorem is that we have a function which is *its own derivative*! That means that the rate of change of e^x at any point is simply e^x evaluated at that point.

Example 2.23. Find the equation of the tangent and normal lines to $y = -e^x$ at $x = 0$.

First, $y(0) = -e^0 = -1 \implies (x_0, y_0) = (0, -1)$.
Now $\frac{\mathrm{d}y}{\mathrm{d}x} = -e^x \implies m = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -e^0 = -1$ and $-\frac{1}{m} = 1$.
So we have the following equations:

Tangent: $y + 1 = -x$ Normal: $y + 1 = x$

Theorem 2.24. If $f(x) = \ln x$, then its derivative is:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

Remember that the domain of $\ln x$ is $(0, \infty)$, so $\ln x$ is not continuous (and indeed not differentiable) for $x \le 0$.

Examples 2.25. 1. If $h(x) = 5\ln x$, then $h'(x) = 5\left(\frac{1}{x}\right) = \frac{5}{x}$

2. If $y = -3\sin x - 4\ln x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = -3\cos x - \frac{4}{x}$
3. Find the instantaneous rate of change of $g(x) = 9\ln x$ at $x = 3$.

   $g'(x) = \frac{9}{x} \implies g'(3) = \frac{9}{3} = 3$

The Product and Quotient Rule

Unfortunately, products and quotients of functions do not act as nicely as sums and differences of functions when differentiating.

Theorem 2.26 (Product Rule). Suppose that f and g are differentiable functions. Then the product fg is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Alternatively, if u and v are expressions in terms of x, then

$$(uv)' = u'v + uv'$$

Example 2.27. Find the derivative of x^3e^x with respect to x.

We take $u = x^3$ and $v = e^x$, from which

$$u' = 3x^2 \quad \text{and} \quad v' = e^x$$

Putting everything together gives us

$$\frac{\mathrm{d}}{\mathrm{d}x}x^3e^x = 3x^2e^x + x^3e^x$$

Unsurprisingly, there is also a rule for differentiating one function divided by another.

Theorem 2.28 (Quotient Rule). Suppose that f and g are differentiable functions. Then the quotient $\frac{f}{g}$ is differentiable whenever $g(x) \neq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Alternatively, if u and v are expressions in terms of x, then

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

You may memorize the product and quotient rule in a similar fashion, but note the minus sign as opposed to the plus in the quotient rule!

Example 2.29. If $f(x) = \frac{1}{1-x}$, find $f'(x)$.

We take $u = 1$ and $v = 1 - x$, from which

$$u' = 0 \quad \text{and} \quad v' = -1$$

Putting everything together gives us

$$f'(x) = \frac{0(1-x) - 1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

Derivatives of Remaining Trigonometric Functions

So far, we have only discussed the derivatives of $\sin x$ and $\cos x$. What about the other four trigonometric functions? Each of them also has their own derivative, but in fact, we can find them by using the or quotient rule!

Theorem 2.30. The trigonometric functions $\tan x$, $\csc x$, $\sec x$, $\cot x$ are differentiable wherever they are defined, with derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$$

Proof. Suppose $f(x) = \sec x = \dfrac{1}{\cos x}$.
We take $u = 1$ and $v = \cos x$, from which

$$u' = 0 \quad \text{and} \quad v' = -\sin x$$

By the quotient rule, we get

$$f'(x) = \frac{0(\cos x) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x$$

The remaining three are left as exercises. ∎

Example 2.31. Find $f'\left(\frac{\pi}{3}\right)$ if $f(x) = x^2 \cot x$.

First, we find $f'(x)$:

$$f'(x) = 2x\cot x + x^2(-\csc^2 x) = 2x \cot x - x^2 \csc^2 x$$

$$\implies f'\left(\frac{\pi}{3}\right) = 2\left(\frac{\pi}{3}\right)\cot\frac{\pi}{3} - \left(\frac{\pi}{3}\right)^2 \csc^2\frac{\pi}{3} = \frac{2\pi}{3}\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi^2}{9}\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{2\pi}{3\sqrt{3}} - \frac{4\pi^2}{27}$$

So far, the derivative rules are simple, but may take plenty of practice to master. Especially when it comes to differentiating using the product and quotient rules, there may be lots of algebra involved. So take your time and write every step of your work, if needed.

Exercises 2.2. 1. Use the quotient rule to *prove* the derivative formulas for $\tan x$, $\cot x$, and $\csc x$.

2. If $y = 5 - x + 2x^2$, find:

   (a) $y(2)$ (b) $\left.\dfrac{dy}{dx}\right|_{x=2}$ (c) $\left.\dfrac{d^2y}{dx^2}\right|_{x=2}$

3. Differentiate the following with respect to x:

   (a) $\sin x \ln x$ (b) $2e^x \tan x$ (c) $\cos^2 x$

4. At what points on $y = x^3 + 9x^2 + 15x - 2$ does the curve have a *horizontal tangent*? (*Hint: What is the slope of a horizontal line?*)

5. The curve $g(x) = 2x^3 + ax + b$ has a tangent with slope 9 at the point $(2, 8)$. Find the values of a and b.

6. Find $\dfrac{dy}{dt}$ for the following functions:

   (a) $y = \dfrac{\sqrt{t}}{t^2 - 3}$ (b) $y = \dfrac{t^3 - t}{t^2}$

7. Consider the curves $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$.

   (a) Sketch the graph of $y = f(x)$ and $y = g(x)$ on the same set of axes. What is the relationship between these two functions?
   (b) Find $f'(x)$ and $g'(x)$.
   (c) Evaluate $f'(2)$ and $g'(8)$. What do you think is the significance of this result?
   (d) Find the equation of the tangent line to $f(x)$ at $x = 2$ and the equation of the tangent line to $g(x)$ at $x = 8$ and sketch them.

8. Suppose the function h is given by

   $$h(x) = \begin{cases} a + bx, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$

   Find constants a, b such that h is differentiable for all x.

9. Let $y = \frac{1}{x^3}$.

   (a) By considering $\frac{1}{x^3}$ as a quotient of the functions 1 and x^3, use the quotient rule to find $\frac{dy}{dx}$.
   (b) By rewriting as $y = x^{-3}$, use the power rule to find $\frac{dy}{dx}$. Which method was easier?

10. Let $f(x) = \sin x$.

    (a) Find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. What do you notice?
    (b) Using your observation in part (a), find $f^{(100)}(x)$.

11. Use the power rule to prove Theorem 2.14

12. (Hard) Use the limit definition of derivative to *prove* the product rule.

2.3 Composite, Implicit, and Inverse Functions

Compound Rates of Change

Suppose that Adam runs at a speed of 4 miles per hour, and that Bob runs twice as fast as Adam. Let the functions $a(t)$ and $b(t)$ represent the distances covered at time t by Adam and Bob, respectively. Then the rate of change of Adam's position *with respect to time* is

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 4$$

However, Bob runs twice as fast, so the rate of change of Bob's position *with respect to Adam's* is

$$\frac{\mathrm{d}b}{\mathrm{d}a} = 2$$

Hopefully this makes sense to you! Remember the language for derivatives 'with respect to □'. What we are saying is that for every 1 hour that passes, Adam travels 4 miles. And for every 1 mile that Adam travels, Bob travels 2 miles.

Now, if we ask at what rate Bob's position is changing *with respect to* t, the solution should be obvious. Twice as fast as Adam means that Bob travels at a rate of 8 miles per hour. The essential observation for this section is that this is the *product* of the two given rates of change:

$$8 = \frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\mathrm{d}b}{\mathrm{d}a} \cdot \frac{\mathrm{d}a}{\mathrm{d}t} = 2 \cdot 4$$

Theorem 2.32 (Chain Rule). Suppose that g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=a} (f \circ g)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=a} f(g(x)) = f'(g(a)) \cdot g'(a)$$

Alternatively, if y is a function of u, and u is a function of x, then in Leibniz's notation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

In Leibniz's notation, it appears that the expressions $\mathrm{d}u$ are being cancelled from top and bottom.[10]

The informal and useful way to think about the chain rule is: for a function which has an 'inside part' and an 'outside part', its derivative is the derivative of the outside part times the derivative of the inside part.

How do we actually use the chain rule in practice? When you encounter a composite function (i.e. a function with an 'inside part' and an 'outside part'), treat them separately differentiating from outside to inside.

I would suggest not labeling each 'part' as f and g or otherwise; you may be likely to make a mistake, and it does take a while. As a practical matter, getting comfortable with differentiating quickly with the chain rule is an important skill.

[10] This is completely unjustified since $\mathrm{d}u$ does not (for us) mean anything on its own! This is merely an intuitive way in which we can think about how the chain rule works.

Examples 2.33. 1. We differentiate $y = \sin(x^2)$ slowly. We'll treat $\sin(\quad)$ as the 'outside part' and x^2 as the 'inside part'.

$$\begin{aligned}
&\textit{Step 1} \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^2) = \cos(\quad)\cdot\frac{d}{dx}(\quad) && \text{(since the derivative of sin is cos)}\\
&\textit{Step 2} \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot\frac{d}{dx}(x^2) && \text{(substitute in 'inside part')}\\
&\textit{Step 3} \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot 2x && \text{(differentiate 'inside part')}\\
&\textit{Step 4} \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^2) = 2x\cos(x^2) && \text{(rearrange for final answer)}
\end{aligned}$$

This is quite a lengthy process, but over time, you will get better at differentiating without needing to write out all these steps! The key skill is identifying the inside and outside parts.

2. If $f(x) = (3x^2 + 4x)^5$, find $f'(x)$.

$$f'(x) = 5(\quad)^4 \cdot \frac{d}{dx}(\quad) = 5(3x^2+4x)^4 \cdot (6x+4) = 5(6x+4)(3x^2+4x)^4$$

The crucial step here was recognizing that $(\quad)^5$ is the outside part and $3x^2 + 4x$ is the inside. The outside part was easily differentiated using the power rule.

3. Find the derivative with respect to x of $y = e^{x^3-x}$.

In this example, we'll use the chain rule in Leibniz's notation. Notice that we can write $y = e^u$, where $u = x^3 - x$. Then we have

$$\frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = 3x^2 - 1 \Longrightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = e^u\cdot(3x^2-1) = (3x^2-1)e^{x^3-x}$$

since $u = x^3 - x$; we simply replaced u with our expression in terms of x (since we want our final answer in terms of x).

4. Calculate $g'(x)$ for $g(x) = 5x^2\cos(2x)$.

Here, we need to use *both* the product rule and chain rule!

$$g'(x) = \underbrace{10x\cdot\cos(2x) + 5x^2\cdot\overbrace{-\sin(2x)\cdot 2}^{\text{chain rule}}}_{\text{product rule}} = 10x\cos(2x) - 10x^2\sin(2x)$$

Be comfortable with applying *multiple* derivative rules per problem.

5. Find the derivative of $y = \ln\left(\tan\left(x^3\right)\right)$

We need to use the chain rule *twice* in this example. Remember, work from outside to inside.

$$\frac{dy}{dx} = \frac{1}{\tan(x^3)}\cdot\sec^2(x^3)\cdot 3x^2$$

Implicit Differentiation

Consider the relation $x^2y = e^{3x}$ and suppose we want to find $\frac{dy}{dx}$, the rate of change of y with respect to x. How can we accomplish this? One idea is to isolate y using algebra, then differentiate as we have before.

$$x^2y = e^{3x} \implies y = \frac{e^{3x}}{x^2} \implies \frac{dy}{dx} = \frac{3e^{3x} \cdot x^2 - e^{3x} \cdot 2x}{x^4} = \frac{e^{3x}(3x-2)}{x^3}$$

However, for *implicit* relations such as $y^3 - 18xy^2 + 7x^2y + 2xy = -13$, it is often extremely difficult or impossible to write y as a function of x. How do we find $\frac{dy}{dx}$ here? We can use the *chain rule*.

Remember that in Leibniz's notation, as per Theorem 2.32, when differentiating an expression with respect to some variable $\Box$, we must multiply by its derivative with respect to $\Box$. So when we have an expression of y, we can differentiate it using our known derivative rules, but then also multiply it by the derivative of y with respect to x, i.e. $\frac{dy}{dx}$. This process is called *implicit differentiation*.

In short, when we find the derivative with respect to x of an implicit relation that has x and y terms, we must multiply by $\frac{dy}{dx}$ each time we differentiate an expression of y, and each time we differentiate an expression of x, we leave as is.

Examples 2.34. 1. Find $\frac{dy}{dx}$ if $x^2 + y^2 = 4$.

$$x^2 + y^2 = 4 \implies \frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(4) \implies 2x + 2y\frac{dy}{dx} = 0$$
$$\implies 2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -\frac{x}{y}$$

Here, we just treated y^2 as a standard term and used the power rule to differentiate. But notice the symbol $\frac{d}{dx}$: if we want to differentiate with respect to x, according to the chain rule, we must multiply each differentiated y term by $\frac{dy}{dx}$.

2. Find $\frac{dy}{dx}$ for $x + x^2y + y^3 = 60$

$$x + x^2y + y^3 = 60 \implies \frac{d}{dx}(x + x^2y + y^3) = \frac{d}{dx}(60) \implies 1 + \underbrace{2xy + x^2\frac{dy}{dx}}_{\text{product rule}} + 3y^2\frac{dy}{dx} = 0$$
$$\implies (x^2 + 3y^2)\frac{dy}{dx} = -1 - 2xy \implies \frac{dy}{dx} = \frac{-1 - 2xy}{x^2 + 3y^2}$$

Again, start getting comfortable with the previous derivative rules.

3. Find the slope of the tangent line to $x^2 + y^3 = 5$ at the point where $x = 2$.

First, we find $\frac{dy}{dx}$: $\quad x^2 + y^3 = 5 \implies 2x + 3y^2 \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{2x}{3y^2}$

On the original curve, when $x = 2$, we have $4 + y^3 = 5 \implies y = 1$

So the slope of the tangent at the point $(2, 1)$ is $\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = -\frac{2(2)}{3(1)^2} = -\frac{4}{3}$

In the last example, notice that we needed to find the corresponding y-value for $x = 2$ on the curve, since our derivative equation had both x and y terms. The notation $\left.\frac{dy}{dx}\right|_{(x,y)=(a,b)}$ simply means 'the derivative of y with respect to x evaluated at the point (a, b)'. Do note that implicit differentiation can also be used to find higher order derivatives!

Example 2.34.3 cont. We will find $\frac{d^2y}{dx^2}$ for $x^2 + y^3 = 5$. We start with $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{2x}{3y^2} \Longrightarrow \frac{d^2y}{dx^2} = -\frac{2 \cdot 3y^2 - 2x \cdot 6y\frac{dy}{dx}}{(3y^2)^2} = -\frac{6y^2 - 12xy\left(-\frac{2x}{3y^2}\right)}{9y^4} = -\frac{8x^2}{9y^5}$$

We used the quotient rule in the first step, and in the second, we can replace $\frac{dy}{dx}$ with $-\frac{2x}{3y^2}$.

Derivatives of General Exponentials

So far we've only considered the derivative of e^x. But what about for general exponential functions?

Theorem 2.35. If $f(x) = b^x$ where $b > 0$, then its derivative is:

$$f'(x) = \frac{d}{dx}b^x = b^x \cdot \ln b$$

Proof. Here are two arguments.

1. (Property of Logarithms) $y = b^x = e^{(\ln b)x} \Longrightarrow \frac{dy}{dx} = e^{(\ln b)x} \cdot \ln b = b^x \cdot \ln b$
2. (Implicit Differentiation) $y = b^x \Longrightarrow \ln y = \ln b^x \Longrightarrow \ln y = x \cdot \ln b$

   $\Longrightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \ln b \Longrightarrow \frac{dy}{dx} = y \cdot \ln b = b^x \cdot \ln b$

■

In either proof, we need to observe that $\ln b$ is in fact a *constant*. So when we differentiate $x \cdot \ln b$, we can treat $\ln b$ as a constant coefficient and simply use the power rule.
Remember that the chain rule also applies when applying this Theorem.

Examples 2.36. 1. If $f(x) = 3^x$, then $f'(x) = 3^x \cdot \ln 3$

2. If $y = 5^{\sin x}$, then $\frac{dy}{dx} = 5^{\sin x} \cdot \ln 5 \cdot \cos x$ and $\left.\frac{dy}{dx}\right|_{x=\pi} = 5^{\sin \pi} \cdot \ln 5 \cdot \cos \pi = -\ln 5$

3. In general, if $y = b^u$ where b is constant and u is a function of x, then $\frac{dy}{dx} = b^u \cdot \ln b \cdot u'$

4. If $y = x^x$, then $\ln y = \ln x^x = x \cdot \ln x \Longrightarrow \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} \Longrightarrow \frac{dy}{dx} = x^x(\ln x + 1)$

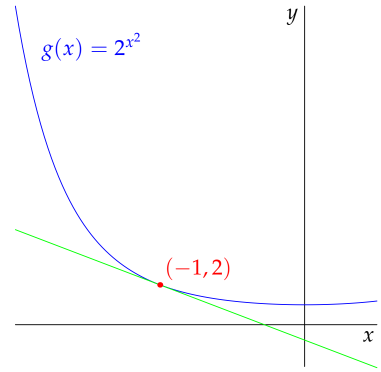
5. Find the equation of the tangent line to $g(x) = 2^{x^2}$ at $x = -1$.

First, $g(-1) = 2^{(-1)^2} = 2$

$g'(x) = 2^{x^2} \cdot \ln 2 \cdot 2x \implies g'(-1) = -4\ln 2$

So the equation of the tangent line is

$$y - 2 = -(4\ln 2)(x+1)$$



Derivatives of Inverse Functions

Recall that, for a function f, its *inverse*, denoted f^{-1}, is the function with unique properties

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \quad \text{and} \quad f(a) = b \iff f^{-1}(b) = a \qquad (*)$$

The first property says that f and f^{-1} *undo* each other, and the second says that the inputs and outputs for f and f^{-1} are *swapped*.

Say that, for $f(x) = x^3 + 1$, we would like to find $(f^{-1})'(2)$, the derivative of the inverse of f evaluated at $x = 2$. One thought is that we can simply find the equation for $f^{-1}(x)$ and differentiate it directly. Remember, to find the inverse of a function, we swap the positions of x and y, then isolate y:

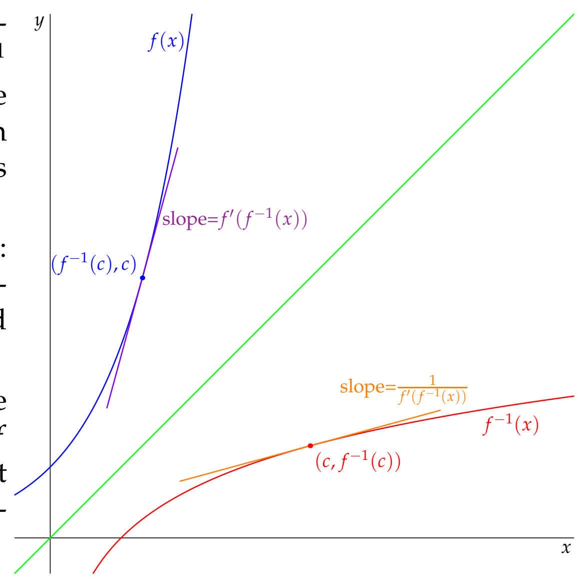
$$y = x^3 + 1 \implies x = y^3 + 1 \implies x - 1 = y^3 \implies y = f^{-1}(x) = (x-1)^{1/3}$$
$$\implies (f^{-1})'(x) = \frac{1}{3}(x-1)^{-2/3} = \frac{1}{3(\sqrt[3]{x-1})^2} \implies (f^{-1})'(2) = \frac{1}{3}$$

It is likely, however, that we encounter functions such as $f(x) = x^3 + 5x + 1$, for which it is impossible to find an explicit formula for $f^{-1}(x)$. How should we find $(f^{-1})'(c)$ for constant c?

We need another approach which does not rely on us explicitly knowing f^{-1}. Recall that the graphs of f and f^{-1} are related by reflection in the line $y = x$; rates of change at corresponding points are therefore *reciprocal*. So we can simply use the slope of the original graph, f' and take its reciprocal.

How do we find the corresponding point? We can use $(*)$: if the point of interest on f^{-1} is $(c, f^{-1}(c))$, then the corresponding point on f is $(f^{-1}(c), c)$, since the inputs and outputs (x- and y-values) are swapped!

As a recap, to find $(f^{-1})'(c)$, we find $f^{-1}(c)$ by finding the x-value for which $f(x) = c$, then find the derivative of f evaluated at $f^{-1}(c)$. We then find the reciprocal of that value. This complicated procedure culminates in the following Theorem.



Theorem 2.37 (Inverse Derivative Rule). If f is invertible with non-zero derivative, then f^{-1} is differentiable and

$$(f^{-1})'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

In Leibniz's notation, if $y = f^{-1}(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$

Invertible means that f indeed does have an inverse; it is one-to-one[11]. We must also specify non-zero derivative because of course the reciprocal of 0, $\frac{1}{0}$, is meaningless.

$\frac{\mathrm{d}x}{\mathrm{d}y}$ is the derivative of x with respect to y, and $\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1}$ is its reciprocal. Think why this makes sense!

Example 2.38. Suppose g is the inverse of $f(x) = x^3 + 2x - 1$. Find $g'(2)$.

From the inverse derivative rule, by letting $g = f^{-1}$, we have

$$g'(x) = \frac{1}{f'(g(x))} \implies g'(2) = \frac{1}{f'(g(2))}$$

To find $g(2)$, we use the fact that inputs and outputs for a function and its inverse are swapped. For some constant b, we have

$$g(2) = b \iff f(b) = 2 \implies b^3 + 2b - 1 = 2$$

By playing around with some values, we easily arrive at $b = 1$. Therefore $g(2) = 1$. We can also find an equation for f':

$$f(x) = x^3 + 2x - 1 \implies f'(x) = 3x^2 + 2 \implies f'(1) = 3(1)^2 + 2 = 5$$

So we end up with

$$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)} = \frac{1}{5}$$

Inverse Trigonometric Functions

We can use what we learned in the previous subsection to differentiate the inverse trigonometric functions. Recall that they can be denoted with the prefix 'arc' or with $^{-1}$ (e.g. the inverse of $\tan x$ is $\tan^{-1} x$ or $\arctan x$).

We try to find the inverse of $y = \arcsin x$ using Leibniz's method. We have

$$y = \arcsin x \iff x = \sin y \implies \frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \implies \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)}$$

[11] In the language of a previous class you may have taken, a function is one-to-one if it passes the vertical and horizontal line test.

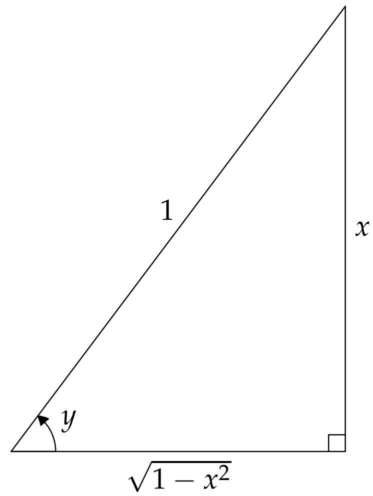
The last expression is very ugly. We can clean it up by remembering that, for $\cos y$, y is an *angle*. Remember also that $x = \sin y$, so we can label an abstract right triangle with the knowledge that

$$\sin y = \frac{x}{1} = \frac{\text{opposite}}{\text{hypotenuse}}$$

Using the Pythagorean Theorem, we see the length of the adjacent leg is $\sqrt{1-x^2}$. Finally,

$$\cos y = \frac{\sqrt{1-x^2}}{1} \implies \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

We could similarly find the derivatives of the remaining inverse trigonometric functions.



Theorem 2.39. The inverse trigonometric functions are differentiable on their domains, with the following derivatives:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

Each of the derivatives of the inverse co-functions is identical to the derivatives of the originals, aside from the minus sign. Remember that all of the previous derivative rules (particularly the chain rule) matter when applying this Theorem!

In general, for AP, the derivatives of the inverse co-functions are slightly less important than the remaining ones.

Examples 2.40. 1. If $f(x) = \sin^{-1}(e^x)$, find $f'(x)$.

$$f'(x) = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

Take care to to multiply by the derivative of e^x, as per the chain rule.

2. Find the slope of the tangent line to $y = \operatorname{arccot}(4x^3)$ at $x = -1$.

$$\frac{dy}{dx} = \frac{-1}{1+(4x^3)^2} \cdot 12x^2 = \frac{-12x^2}{1+16x^6} \implies \left.\frac{dy}{dx}\right|_{x=-1} = \frac{-12\,(-1)^2}{1+16\,(-1)^6} = -\frac{12}{17}$$

3. Differentiate $g(x) = \operatorname{arcsec}(x^2)$ with respect to x.

$$g'(x) = \frac{1}{|x^2|\sqrt{(x^2)^2-1}} \cdot 2x = \frac{2x}{x^2\sqrt{x^4-1}} = \frac{2}{x\sqrt{x^4-1}}$$

We can remove the absolute value symbols at the second step since x^2 is always positive.

4. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $\tan^{-1}(xy) = 3x$.

   We implicitly differentiate:

$$\underbrace{\frac{1}{1+(xy)^2}\overbrace{\left(1\cdot y + x\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)}^{\text{product rule}}}_{\text{chain rule}} = 3 \implies y + x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 3\left(1+x^2y^2\right) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3+3x^2y^2-y}{x}$$

Aside: what method to use? By now, we've learned all of the derivative rules. It can be very daunting to try to have everything memorized and drilled; the only way to do so is, as said, practice! One other thing is that you should be smart about which derivative rule to apply. For example, if $f(x) = 2\sin x\cos x$, we can find $f'(x)$ in several ways:

$$f'(x) = 2\cos x\cdot\cos x + 2\sin x\cdot -\sin x = 2\cos 2x = 2\cos^2 x - 2\sin^2 x = 2\cos 2x$$

or

$$f(x) = 2\sin 2x \implies f'(x) = 2\cos 2x$$

In the first method, we noticed that $2\sin x\cos x$ is a product and thus used the product rule to find f'. In the second, we simply used an identity to simplify, then use the chain rule to differentiate. The second method is much easier in this scenario, but in others, it may be faster to do the opposite.

As another example, say we want to differentiate $g(x) = \frac{x^3-5}{x^2}$. There are a few ways to do this:

$$g'(x) = \frac{3x^2\cdot x^2 - (x^3-5)\cdot 2x}{x^4} = \frac{3x^4-2x^4+10x}{x^4} = 1+\frac{10}{x^3}$$

or

$$g(x) = x - 5x^{-2} \implies g'(x) = 1 + \frac{10}{x^3}$$

The second method is much easier! Recognizing that you can simplify before differentiating may save a lot of time and prevent errors. But in either case, we ended up with the same derivative.

Exercises 2.3. 1. Find the derivatives with respect to x of the following functions:

(a) $-\ln(\cos x)$ (b) $\csc(-4x)$ (c) $12\left(3x-7x^3+14\right)^5$

(d) $e^{\sin x}$ (e) $\cos^{-1}(2x)$ (f) $\sqrt[3]{2x^3-x^2}$

(g) $\dfrac{x}{\sqrt{\sec x}}$ (h) $\ln\left(\dfrac{(x+2)^3}{x}\right)$ (i) $\cos\left(e^{x^5-3x}\right)$

2. Give the coordinates of all points on the curve $\left(x^2+y^2\right)^2=x^2-y^2$ which have a *vertical tangent*. (*Hint: What is the 'slope' of a vertical line?*)

3. Find the derivative of $f(x)=\sin(\sin(\sin(\sin x)))$.

4. Selected values for differentiable functions f, g, and their derivatives are shown in the table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	2	-4	5	0
2	5	6	-1	3
5	-3	12	3	10

(a) If $h(x)=f(g(x))$, find $h'(2)$.
(b) If $k(x)=f(x)g(x)$, find $k'(5)$.

5. Given $y=(1-\frac{1}{3}x)^3$, show that $\dfrac{d^3y}{dx^3}=-\dfrac{2}{9}$

6. Find $(f^{-1})'(3)$ for $f(x)=x^3-2x^2+5x-1$ without using a calculator.

7. Consider the relation $3x^2+2y=2xy$.

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$.

(b) What do you notice? Why does this make sense?

8. For each of the following implicit relations, find $\dfrac{d^2y}{dx^2}$.

(a) $x^2+y^2=25$ (b) $x^2-y^2=10$ (c) $x^3+2xy=4$

9. By using a similar method to what we used in the proof of Theorem 2.35, find the derivative of $y=\log_2 x$.

10. Find the equation of the tangent and normal line to $y=5^{\sqrt{x}}$ at the point where $x=4$.

11. Use an argument similar to the one preceding Theorem 2.39 to prove that

$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$

12. Evaluate the following limits.

(Hint: Don't try to calculate these manually! Recall the limit definition of derivative...)

(a) $\lim_{h \to 0} \frac{\sec(2x+2h) - \sec(2x)}{h}$ (b) $\lim_{x \to e} \frac{\ln x - 1}{x - e}$

13. Show that every normal line to the unit circle $x^2 + y^2 = 1$ passes through the origin.

14. (Hard) Use the power rule, product rule, and chain rule to *prove* the quotient rule.

3 Applications of Differentiation

3.1 Properties of Curves

Now that we are able to quickly differentiate functions, we can now tackle the applications of derivatives. First, let's discuss a property of continuous functions.

Theorem 3.1 (Extreme Value Theorem). If a function f is continuous on an interval $[a, b]$, then f must attain its bounds on that interval. Otherwise said, f must have a minimum and a maximum on the interval. In particular, the minimum and maximum are *finite*.

This Theorem might seem trivial, and without having developed rigorous analysis, it is difficult to prove.

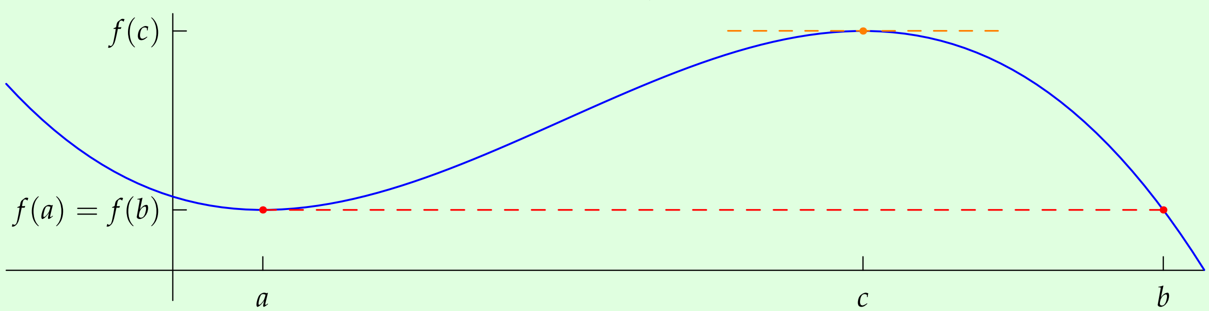
One way you may think about it is that if f is continuous on $[a, b]$, then it certainly does not have an infinite discontinuity. So all values of f on $[a, b]$ must be bounded.

We'll return to this later, but we will consider another Theorem first.

Theorem 3.2 (Rolle's Theorem). Suppose a function f satisfies the following conditions:

1. f is *continuous* on the *closed* interval $[a, b]$
2. f is *differentiable* on the *open* interval (a, b)
3. $f(a) = f(b)$

Then there is some value c between a and b such that $f'(c) = 0$.



The essential idea is that if a differentiable function starts and finishes at the same y-value, and starts heading upwards, then at some point it must turn around and start heading downward again (or vice versa).

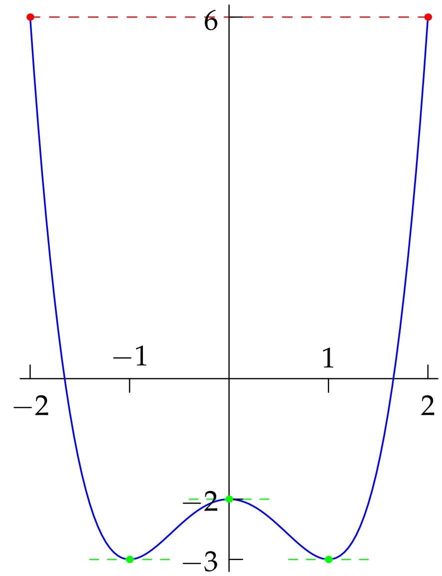
A more concise way of putting it is that if the average slope[12]of f between $x = a$ and $x = b$ equals 0 (since $f(b) - f(a) = 0$), then at some x-value c on the interval $[a, b]$, the instantaneous slope of f i.e. the *derivative* of f evaluated at $x = c$, $f'(c)$, equals the average slope.

We will not make an attempt to prove Rolle's Theorem, but this along with the Extreme Value Theorem (EVT) leads us to the more general result.

Similarly to IVT, there may be more than one choice of c, but Rolle's Theorem only guarantees at least one.

[12]Remember that the average slope of a function f over an interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$

Example 3.3. Consider the function $g(x) = x^4 - 2x^2 - 2$ on the interval $[-2, 2]$. Verify that g satisfies the hypotheses (conditions) of Rolle's Theorem on the given interval and find all values of c guaranteed by the conclusion of Rolle's Theorem.



g is a polynomial, so it is continuous on $[-2, 2]$.

g' is also a polynomial, so g is differentiable on $(-2, 2)$.

$g(-2) = g(2) = 6$

Now to find the values of c:

$$\begin{aligned} g'(x) =& 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1) = 0 \\ &\implies c_1 = -1, c_2 = 0, c_3 = 1 \end{aligned}$$

In this example, there were *three* choices of c.

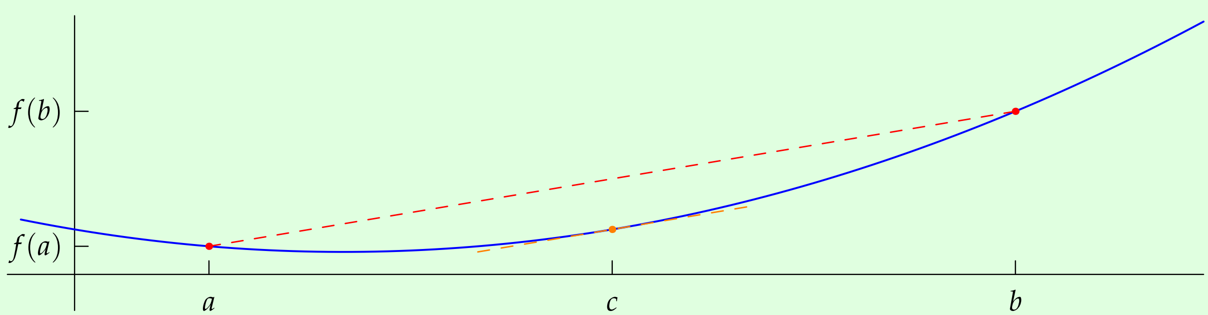
Now we turn to a stronger result.

Theorem 3.4 (Mean Value Theorem). Suppose a function f satisfies the following conditions:

1. f is *continuous* on the *closed* interval $[a, b]$
2. f is *differentiable* on the *open* interval (a, b)

Then there is some value c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



There seems to be a striking similarity between Rolle's Theorem and Mean Value Theorem (MVT); indeed, Rolle's Theorem is a special case of MVT.

MVT tells us that if a function is smooth on an interval, then the average rate of change will equal the derivative somewhere on the interval.

As before, there may be multiple values of c which satisfy the conclusion of MVT, but only one is guaranteed.

When doing any problems related to MVT or Rolle's Theorem, make sure you check the conditions. If either fails, then MVT/Rolle's does not apply.

Examples 3.5. 1. Find all the possible values $x = c$ satisfying the Mean Value Theorem on the interval $[-2, 6]$ for $f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2 + x$.

f is a polynomial, so it is continuous on $[-2, 6]$.

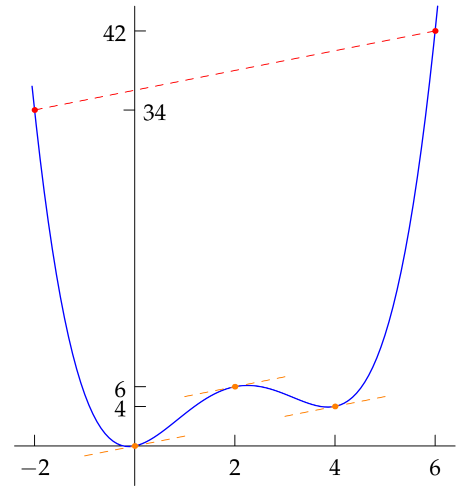
f' is a polynomial, so f is differentiable on $(-2, 6)$.

We find the *average slope* of f on $[-2, 6]$:

$$\frac{f(6) - f(-2)}{6 - (-2)} = \frac{42 - 34}{8} = 1$$

We also find $f'(x) = x^3 - 6x^2 + 8x + 1$. To find c:

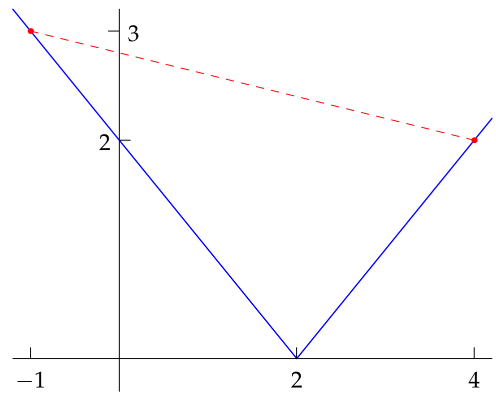
$$c^3 - 6c^2 + 8c + 1 = 1 \implies c^3 - 6c^2 + 8c = 0$$
$$\implies c(c-2)(c-4) = 0 \implies c = 0, 2, \text{or } 4$$



2. If applicable, find all values $x = c$ satisfying MVT on the interval $[-1, 4]$ for $g(x) = |x - 2|$.

   g is continuous on $[-1, 4]$, as shown by the graph.
   g is *not* differentiable at $x = 2$; it has a *corner*.

   As a result, MVT is not applicable to g on any interval containing $x = 2$. Hopefully this is reasonable in a visual sense as well; if the graph of g has a sharp turn, then there is no guarantee that the slope of the tangent behaves nicely.



3. Find all values c satisfying MVT for $h(x) = \frac{1}{x}$ on the interval $[1, 5]$, if applicable.

   $h(x) = x^{-1}$ has an infinite discontinuity at $x = 0$. However, this x-value is not part of the interval we are concerned with, $[1, 5]$, so the first hypothesis is satisfied!

   $h'(x) = -x^{-2}$ is similarly undefined at $x = 0$, so h is non-differentiable at $x = 0$. No matter! We only care about the interval $(1, 5)$, so the second condition is met.

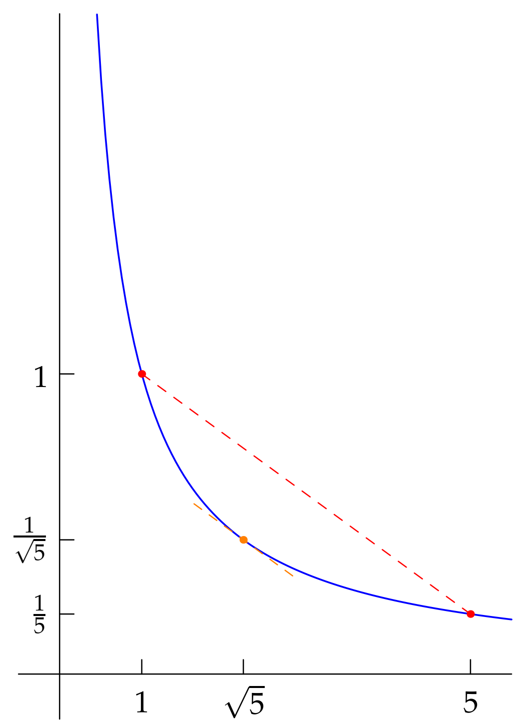
   We find the *average slope* of h on $[1, 5]$:

$$\frac{h(5) - h(1)}{5 - 1} = \frac{\frac{1}{5} - 1}{4} = -\frac{1}{5}$$

   Also $h'(x) = -\frac{1}{x^2}$, so to find c:

$$-\frac{1}{c^2} = -\frac{1}{5} \implies c^2 = 5 \implies c = \pm\sqrt{5}$$

   But $c = -\sqrt{5}$ is outside of our interval of interest. So our only answer is $c = \sqrt{5}$.



Behavior of a Graph

It is worthwhile to revisit a concept from a previous course.

Definition 3.6. Suppose a function f is defined on an interval I. We say that f is:

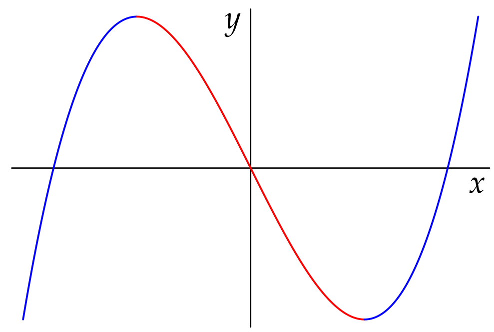
Increasing on I if $a < b$ implies $f(a) \le f(b)$

Decreasing on I if $a < b$ implies $f(a) \ge f(b)$

All of this confusing language is to say that a function f is increasing if, as you trace the graph of f going from left to right, the graph is going up.

Similarly, f is decreasing if, as you trace the graph of f going from left to right, the graph is going down

As for the points 'between' the increasing and decreasing intervals, we say in this course that f is neither decreasing nor increasing.



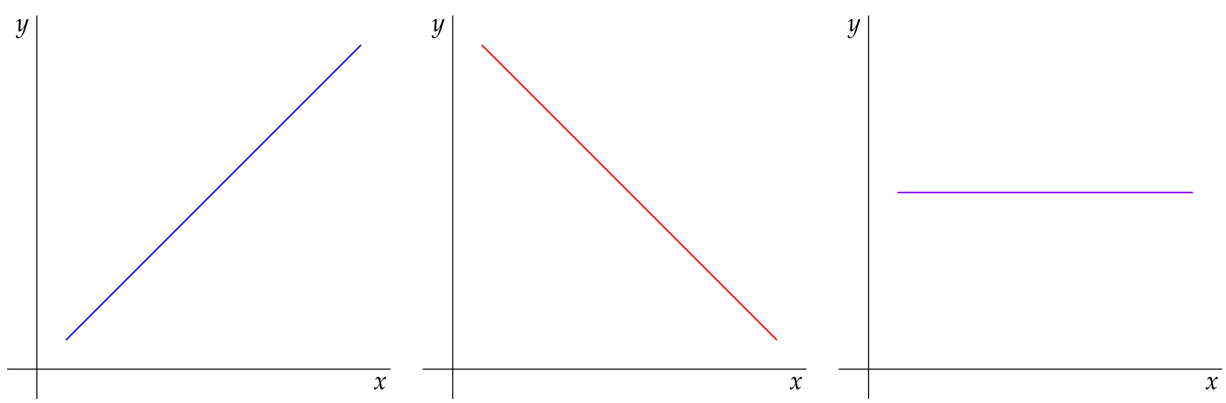
The points 'between' are worth special consideration as well.

Definition 3.7. $f(c)$ is a *local/relative maximum* of f if $f(c) \ge f(x)$ for all x near c. That is, f changes from increasing to decreasing at the point $(c, f(c))$.

Similarly, $f(c)$ is a *local/relative minimum* of f if $f(c) \le f(x)$ for all x near c. That is, f changes from decreasing to increasing at the point $(c, f(c))$.

The rigorous definition of 'near' when talking about local extrema is difficult and requires a discussion beyond the level of this course. Hopefully each of these aforementioned topics are not too foreign; they should have been discussed in a prerequisite course at some point.

How do each of these definitions relate to *derivatives*? It may be useful to return to the most elementary example as we did previously: linear functions. Recall that a line with a *positive* slope is *increasing*. A line with a *negative* slope is *decreasing*. And a line with a slope of 0 is *neither increasing nor decreasing*.



Increasing $\iff$ $m > 0$ Decreasing $\iff$ $m < 0$ Constant $\iff$ $m = 0$

These observations lead us to the following result.

Theorem 3.8. Suppose a function f is differentiable on an interval I. Then:[13]

f is increasing on $I \iff f'(x) \geq 0$ on I

f is decreasing on $I \iff f'(x) \leq 0$ on I

f is constant on $I \iff f'(x) = 0$ on I

Part of the above Theorem follows from MVT.

The key to understanding this Theorem is remembering that derivatives represent slope of the tangent line at any x-value. If the derivative $f'(x)$ is positive, then the *rate of change* of y with respect to x (i.e. $\frac{\Delta y}{\Delta x}$) must be positive. So as the x-value goes up (we travel from left to right), the y-value must be going up. A similar argument works for the decreasing case. And if $f'(x) = 0$, then the value of $f(x)$ must not be changing, i.e. is constant.

Before we get to the result involving relative extrema, we'll look at a quick definition.

Definition 3.9 (Critical Points). Let f be a function. We say that $x = c$ is a *critical value* of f if the derivative $f'(c) = 0$ or is undefined. We call the point with coordinate $(c, f(c))$ a *critical point* of f.

We use this definition in the following Theorem.

Theorem 3.10 (First Derivative Test). Suppose f is continuous on an interval I containing $x = c$ and differentiable except perhaps at c, and that $x = c$ is a critical value of f. Then one of the following must occur:

If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at $x = c$. Otherwise said, if $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a local maximum of f.

If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local maximum at $x = c$. Otherwise said, if $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a local minimum of f.

If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is neither a local maximum nor minimum of f.

In general, every relative extreme is a critical point of f, but not every critical point of f is a relative extreme.

This result follows directly from Definition 3.7 and Theorem 3.8. Read them and understand exactly why the above result makes sense!

Why is every local extreme a critical point? Observe that at a local extreme, f' must change signs at $x = c$. Since f is differentiable, then by IVT, the graph of f' must pass through $y = 0$ at $x = c$ for it to change signs, which is the very definition of a critical value! Otherwise, if f is not differentiable with continuous derivative at $x = c$, the graph must break off to change signs, making f' undefined at $x = c$, which also makes it a critical point.

[13] The symbol $\iff$ is read 'if and only if' or 'is equivalent to'. It means that the conditional goes both ways, i.e. if f is increasing, then $f'(x) > 0$ *and* if $f'(x) > 0$, then f is increasing. Take an undergraduate college course in logic if you want to know more!

A very common type of problem we will encounter is one in which we are asked to find the intervals for which a function is increasing or decreasing, and to find the relative extrema of a function f. This is the basic method:

1. Find $f'(x)$ and set $f'(x) = 0$ in order to find the *critical values*. Remember to also consider the undefined cases.
2. Construct a *first derivative sign chart* (shown in the following examples) and label the critical values. Since f' is continuous, it cannot change signs without passing through a $y = 0$ by IVT.
3. Test a value in each interval between critical values to check whether f' is positive or negative. Label these on the sign chart.
4. Classify each critical value as a relative maximum, minimum, or neither by observing the sign chart. If required, find the relative maximum/minimum *values* by substituting into $f(x)$.
5. If asked to justify your answer, refer directly to the Definitions and Theorems.

If a problem asks for the *location* of any local extrema, simply give the x-values. If asked for the relative minimum or maximum *values* or simply relative minima or maxima, give the y-values (i.e. f(critical values)). If asked for the relative minimum/maximum points, give the coordinates.

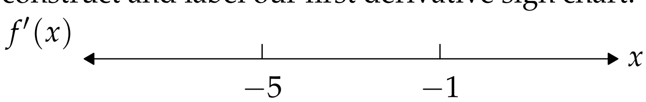
Perform any combination of these steps to answer the question fully and clearly.

Examples 3.11. 1. Find all intervals for which the function $f(x) = x^3 + 9x^2 + 15x - 1$ is increasing or decreasing. Also identify and classify all critical values of f.

First, we find all critical values:

$$f'(x) = 3x^2 + 18x + 15 = 3(x+1)(x+5) = 0$$

So our critical values are $x = -1$ and $x = -5$. We construct and label our first derivative sign chart:



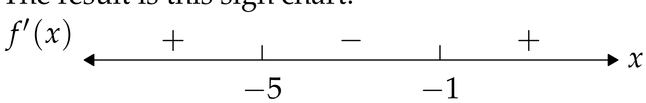
We test several values in each interval.

$f'(-6) = 15 > 0$, so we label '+' above.

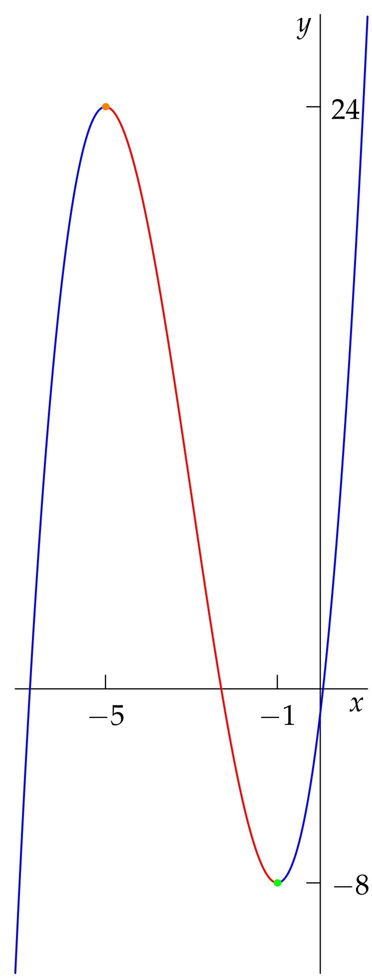
$f'(-2) = -9 < 0$, so we label '−' above.

$f'(0) = 15 > 0$, so we label '+' above.

The result is this sign chart:



f is increasing on intervals $(-\infty, -5)$ and $(-1, \infty)$.
f is decreasing on the interval $(-5, -1)$.
f has a local maximum at $x = -5$
f has a local minimum at $x = -1$

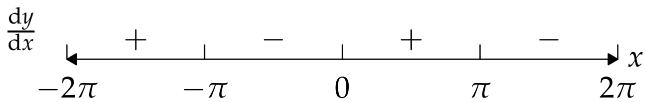


2. Find all intervals for which $y = -\cos x$ is increasing or decreasing. Also identify and classify all *critical points* of f. Justify your answer in each case.

   First, we find all the critical values:

   $$\frac{dy}{dx} = \sin x = 0 \Longrightarrow x = 0, \pm\pi, \pm 2\pi, \cdots = n\pi \text{ for any integer } n.$$

   As it turns out, $y = -\cos x$ has a critical value at every integer multiple of π! We only need to make our sign chart for one cycle (since $\sin x$ is periodic) then fill it in. Remember, we are testing values for the *derivative* $\frac{dy}{dx} = \sin x$, not the original function $y = -\cos x$:

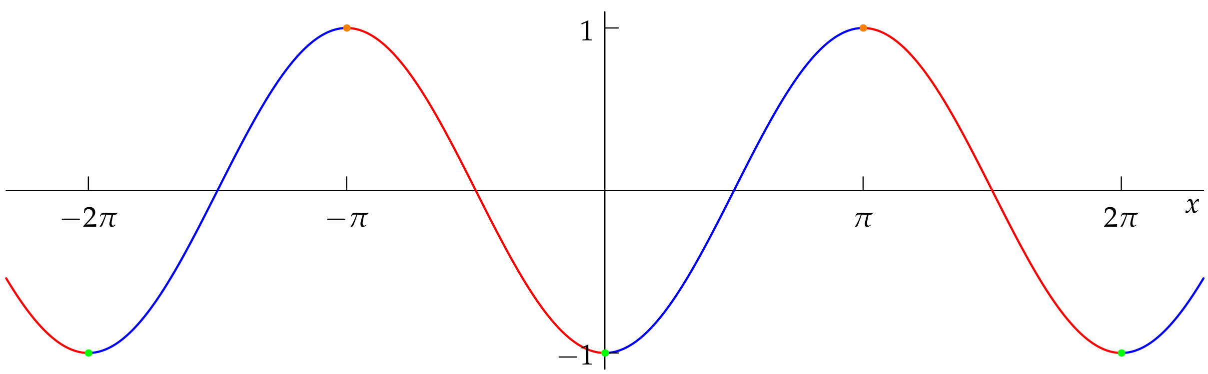
   

   $y = -\cos x$ is increasing on intervals $(0, \pi)$, $(2\pi, 3\pi)$, ..., which can also be written as $n\pi < x < (n+1)\pi$ for any *even* integer n, since on those intervals, $\frac{dy}{dx} > 0$.
   $y = -\cos x$ is decreasing on intervals $(-\pi, 0)$, $(\pi, 2\pi)$, ..., which can also be written as $n\pi < x < (n+1)\pi$ for any *odd* integer n, since on those intervals, $\frac{dy}{dx} > 0$.
   $y = -\cos x$ has a local maximum at $x = -\pi, \pi, 3\pi, \ldots$, or at every odd integer multiple of π, since at those x-values, $\frac{dy}{dx}$ changes from positive to negative. The relative maximum *points* are $(-\pi, 1)$, $(\pi, 1)$, etc.
   $y = -\cos x$ has a local minimum at $x = -2\pi, 0, 2\pi, \ldots$, or at every even integer multiple of π, since at those x-values, $\frac{dy}{dx}$ changes from negative to positive. The relative minimum *points* are $(0, -1)$, $(2\pi, -1)$, etc.

   This is quite a confusing example, but notice first how we justified each of our answers by directly quoting the Theorems. This is all you need to say to explain your answer!

   Also notice that the question directed us to provide the critical *points* of the function, so we needed to take each of the x-values for the respective part and substitute them into the *original function* $y = -\cos x$, not the derivative! We then gave the *coordinates* of the local maximum and minimum points. Try not to get confused between intervals and coordinate notation!

   A graph of $y = -\cos x$ is provided here. Notice that the periodicity of $-\cos x$ is reflected in the derivative, causing the graph of y to have infinitely many minimum and maximum points, as well as infinitely many intervals for which y is increasing or decreasing.

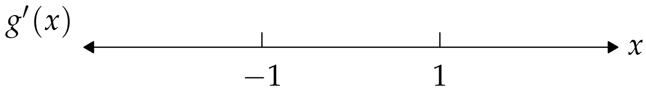
   

3. Find all relative extrema of $g(x) = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 - x$, and find all intervals for which g is increasing or decreasing. Justify your answers.

   First, we find all the critical values:

   $$g'(x) = -x^3 + x^2 + x - 1 = -x^2(x-1) + (x-1)\text{[14]} = (1-x^2)(x-1)$$
   $$= -(x-1)^2(x+1) = 0 \Longrightarrow x = -1, 1$$

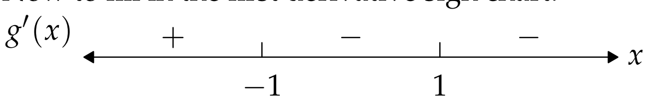
   So our critical values are $x = -1$ and $x = 1$. We draw a sign chart and label the critical values:



   One neat trick to avoid nasty calculations when testing values in each interval for g' is to substitute into the *factored* form and only mind the signs. Remember, we only care about whether g' is positive or negative!

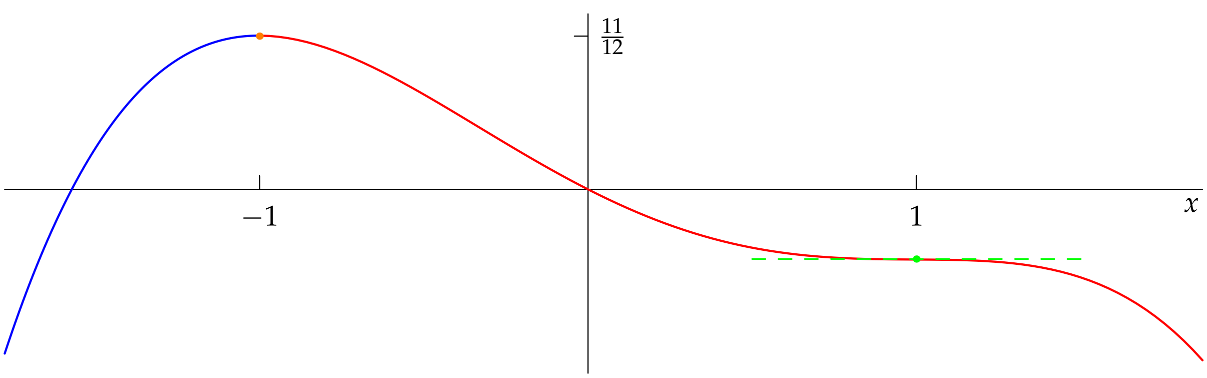
   $$g'(-2) = -(-2-1)^2(-2+1) = -(-)^2(-) = -(+)(-) = +$$
   $$g'(0) = -(0-1)^2(0+1) = -(-)^2(+) = -(+)(+) = -$$
   $$g'(2) = -(2-1)^2(2+1) = -(+)^2(+) = -(+)(+) = -$$

   Now to fill in the first derivative sign chart:



   g is increasing on the interval $(-\infty, -1)$, since $g'(x) > 0$ on this interval.
   g is decreasing on the intervals $(-1, 1)$ and $(1, \infty)$, since $g'(x) < 0$ on these intervals.
   g has a relative maximum at $x = -1 \Longrightarrow g(-1) = \frac{11}{12}$, since $g'(x)$ changes from positive to negative at this value.
   g has *no relative minima*, since $g'(x)$ does not change from negative to positive anywhere.

   Note that $g'(x)$ did not change signs at $x = 1$. This is an example of a critical point which is *not* a local extreme. The graph of g flattens out (has a horizontal tangent) at $x = 1$, but continues moving downward after. Also note that we said that g is decreasing on $(-1, 1)$ and $(1, \infty)$ rather than $(-1, \infty)$, since g is neither decreasing nor increasing at $x = 1$; it is stationary!



[14] We used *factoring by grouping* here. Remembering algebraic methods is very important!

Absolute Extrema

We now know how to identify *local* extrema, but we also need to be able to find *global* extrema.

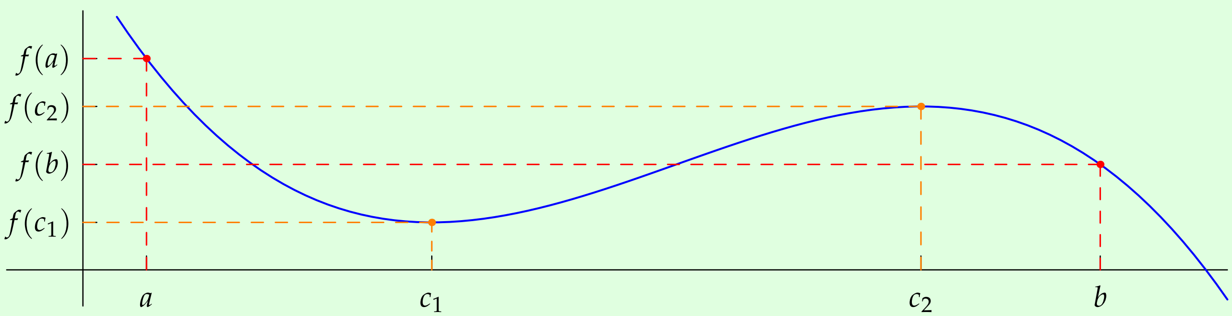
Definition 3.12. $f(c)$ is the *(absolute) maximum (value)* of f on $[a, b]$ if $f(c) \geq f(x)$ for all x in $[a, b]$. The point $(c, f(c))$ is called the *absolute maximum point*.

Similarly, $f(c)$ is the *(absolute) minimum (value)* of f on $[a, b]$ if $f(c) \leq f(x)$ for all x in $[a, b]$. The point $(c, f(c))$ is called the *absolute minimum point*.

The absolute maximum and minimum for a function on an interval are, respectively, the highest and lowest points on that interval.

When we are asked to find absolute extrema, we must always be provided with an *interval*. Asking for absolute extrema over all real numbers for an unbounded function is meaningless!

Theorem 3.13. The absolute maximum and minimum of a differentiable function f on the interval $[a, b]$ must occur at the *relative extrema* $x = c_n$ or the *endpoints* of the interval, i.e. $x = a$ or $x = b$.



In the above picture, the absolute maximum occurred at $x = a$, which is one of the endpoints. The absolute minimum occurred at $x = c_1$, which is one of the critical values. The takeaway is if we are asked to find absolute extrema, we must make sure to check *both* endpoints and all critical values! Here is the strategy for finding absolute extrema of a function:

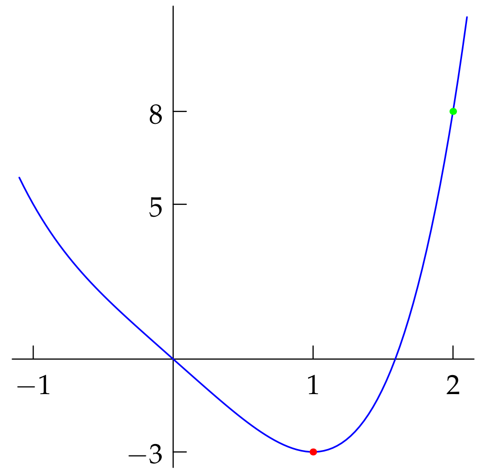
1. Find the critical values by setting $f'(x) = 0$.
2. Construct a table of values for $f(x)$ (not $f'(x)$!), including the critical values you found as well as the endpoints, then fill it in.
3. Mind the language used in the question—if asked for the *location* of the absolute extrema, give the respective x-values. If asked for the *maximum/minimum, absolute maximum/minimum, or maximum/minimum value* of f on $[a, b]$, give the respective y-value ($f(x)$ value). If asked for the absolute maximum/minimum *points*, give the respective coordinate.

It is unnecessary to make a first derivative sign chart unless the question also instructs us to classify the critical points.

The reason we are inputting values into f rather than f' is because we are trying to find the extrema of f! Inputting into f' does not help us with this.

You may also be asked to justify your answer when finding absolute extrema. The table of values suffices!

Examples 3.14. 1. Find the absolute minimum and maximum of $f(x) = x^4 - 4x$ on $[-1, 2]$. Justify your answer.



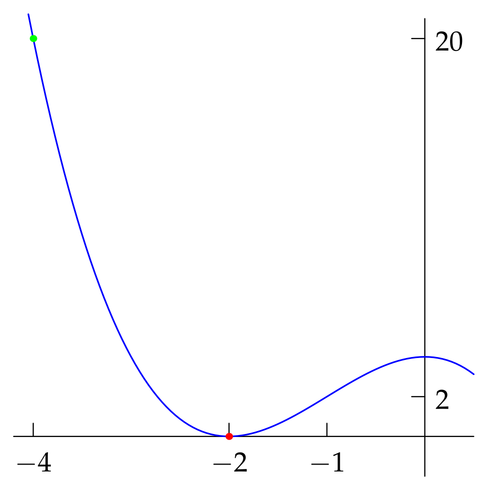
$$f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 0 \implies x = 1$$

Our only critical value is $x = 1$. We construct a table and fill it in by inputting the selected values into $f(x)$:

x	-1	1	2
$f(x)$	5	-3	8

From the table, we see that, on $[-1, 2]$, the maximum value of f is $f(2) = 8$. The minimum is $f(1) = -3$.

2. Find the location of the absolute minimum and maximum of $g(x) = -x^3 - 3x^2 + 4$ on $[-4, -1]$.



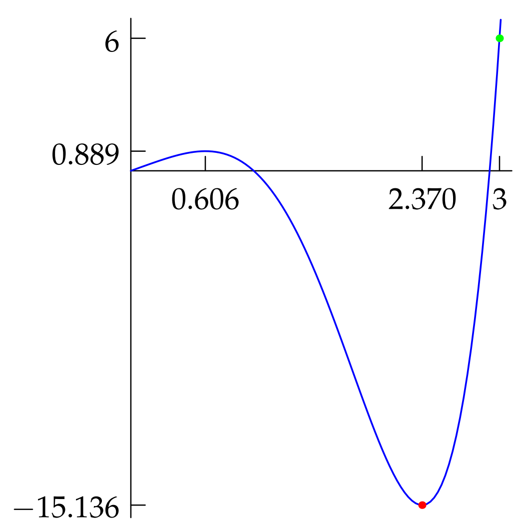
$$g'(x) = -3x^2 - 6x = -3x(x + 2) = 0$$
$$\implies x = -2, 0$$

$x = 0$ does not lie in $[-4, -1]$, so we need not include it on our table of values:

x	-4	-2	-1
$g(x)$	20	0	2

From the table, we see that the absolute maximum of g on $[-4, -1]$ occurs at $x = -4$. The absolute minimum occurs at $x = -2$.

3. (Calculator) Find the absolute minimum and maximum points of $y = x^5 - 3x^4 + 2x$ for $0 \le x \le 3$.



On our calculators, we use the 'zero' function to find the solutions of:

$$Y_1 = \frac{\mathrm{d}}{\mathrm{d}x}(x^5 - 3x^4 + 2x)\Big|_{x=x} = 0 \text{ on } [0, 3]$$

From which we get $x = 0.606$ and $x = 2.37$. Now for the table:

x	0	0.606	2.370	3
y	0	0.889	-15.136	6

Thus, on $[0, 3]$, the maximum point is $(3, 6)$, and the minimum point is $(2.370, -15.136)$.

Exercises 3.1. 1. Check that $f(x) = (1 + 3x^2)^{-1/2}$ for $-1 \leq x \leq 1$ satisfies the hypotheses of Rolle's Theorem and find all values $x = c$ guaranteed by its conclusion.

2. For the following functions, if applicable, find all values $x = c$ which satisfy the conclusion of the Mean Value Theorem.
   (a) $f(x) = 9 - x^2$ on $[-2, 3]$
   (b) $g(x) = x + \sin x$ on $[0, 3\pi]$
   (c) $h(x) = x^{2/3} - 3$ on $[-8, 8]$

3. Try to give an example of a function f which is continuous on some interval $[a, b]$ but not differentiable on (a, b), and hence fails to satisfy the hypotheses of the Mean Value Theorem.

4. For the following functions f, find the intervals for which f is increasing, decreasing, and classify all critical points of f.
   (a) $f(x) = e^{x^2 - 3x}$
   (b) $f(x) = x^3 - 3x^2 - 9x + 5$
   (c) $f(x) = x^2 e^x$
   (d) $f(x) = x - \sqrt{x}$

5. Recall that the *standard form* for a quadratic is $y = ax^2 + bx + c$ for constants a, b, c and $a \neq 0$.
   (a) Explain why the *vertex* of the parabola must be located at the point where $x = -\frac{b}{2a}$.
   (b) Repeat for the *vertex form* $y = a(x - h)^2 + k$ to show that the x-value of the vertex is h.
   (c) Under what conditions is the vertex a local minimum or a local maximum?

6. $g(x) = x^3 + ax + b$ has a critical point at $(-2, 3)$. Find the values of a and b.

7. Find the absolute minima and maxima of the following functions on the given interval.
   (a) $f(x) = \sin 2x + 2\cos x$ on $[0, \frac{3\pi}{2}]$
   (b) $g(x) = x^3 - 12x - 2$ on $-3 \leq x \leq 5$
   (c) (Calculator) $h(x) = \frac{1}{x} \ln x$ on $[\frac{1}{2}, 5]$

8. Show that the function $y = x - \ln x$ has a local minimum and that this is the *only* critical point.

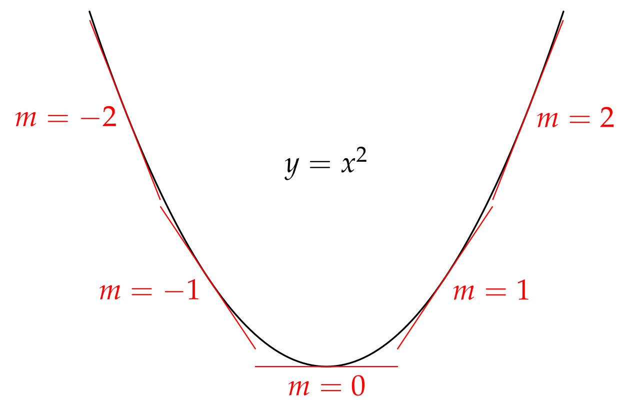
9. Explain why $x = -\frac{1}{\sqrt{2}}$ is *not* a critical value of $f(x) = \ln x - x^2$.

10. Show that $g(x) = 4e^{-x} \sin x$ has a critical value at $x = \frac{\pi}{4}$. Is this a local minimum, local maximum, or neither?

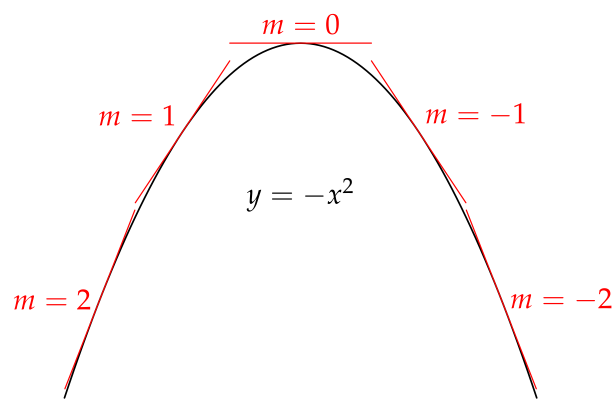
11. (Hard) Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.
    (a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c) = -1$.
    (b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for which $2 < k < 5$ such that $g''(k) = 0$.
    (c) Let $h(x) = f(x) - x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r) = 0$.

3.2 Shape, Inflections, and Derivative Relationships

While the relationship between a function and its derivative is visually clear, what about a function and its *second derivative*? Conceptually speaking, for a function f, its second derivative f'' is the first derivative of its first derivative, f'; f'' is therefore the rate of change *of the rate of change*. How can we visualize this? Consider the curves $y = x^2$ and $y = -x^2$:



Wherever we are on the curve, as x increases, the slope of the tangent *increases*.
Thus $f'(x)$ increasing $\iff f''(x) > 0$



Wherever we are on the curve, as x increases, the slope of the tangent *decreases*.
Thus $f'(x)$ decreasing $\iff f''(x) < 0$

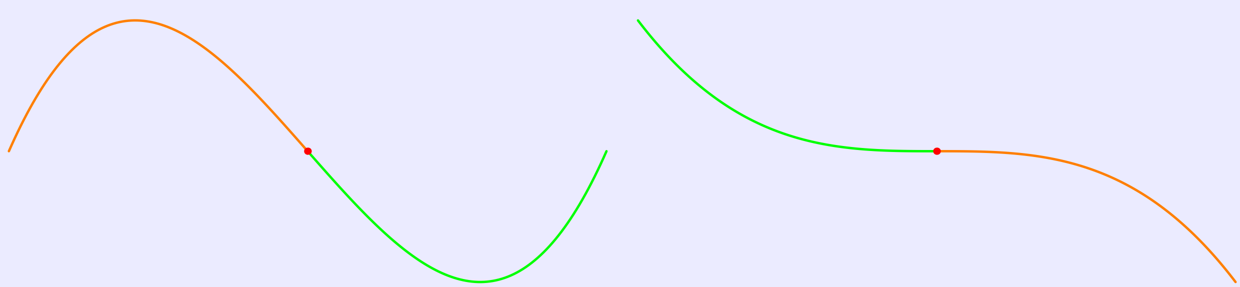
These observations allow us to define the notion of *concavity*.

Definition 3.15 (Concavity). Suppose a function f is twice-differentiable on an interval I. Then:

f is *concave up* on $I \iff f''(x) > 0$ on I

f is *concave down* on $I \iff f''(x) < 0$ on I

A *point of inflection* of f is a point on the curve at which there is a change of concavity.



Concavity of a curve refers to its *shape*, or *curvature*[15]. A curve defined by f is concave up (opening upward) when it second derivative f'' is positive, and concave down (opening downward) when f'' is negative.

The function f' also has critical values. They occur at $x = c$ such that $f''(c) = 0$ or is undefined. Perhaps unsurprisingly, the critical values of f' relate to the points of inflection of f.

[15]The strict notion of curvature requires development of differential geometry, so just stick with the informal visualization for now... One potential way to remember concavity is ”concave up like a ’cup’, concave down like a ’frown’”!

Theorem 3.16. Suppose f is continuous on an interval I containing $x = c$ and twice-differentiable except perhaps at c, and that $x = c$ is a critical value of f'. Then one of the following must occur:

If $f''(x) > 0$ for $x < c$ and $f''(x) < 0$ for $x > c$ or $f''(x) < 0$ for $x < c$ and $f''(x) > 0$ for $x > c$, then f has a point of inflection at $x = c$. Otherwise said, if $f''(x)$ changes sign at $x = c$, then $(c, f(c))$ is a point of inflection of f.

If $f''(x)$ does not change sign at $x = c$, then $(c, f(c))$ is not a point of inflection of f.

In general, every point of inflection of f is a critical point of f', but not every critical point of f' is a point of inflection of f.

Notice the difference in language; a critical value of f is *not* the same as a critical value of f'! A critical value of f occurs when $f'(x) = 0$ or is undefined, and a critical value of f' occurs when $f''(x) = 0$ or is undefined.

Similarly to the previous section, a common question we will be asked is to find the intervals for which a function f is concave down or concave up, and to find the points of inflection of f. The method is identical to the one we used for finding intervals for f increasing/decreasing and locating relative extrema:

1. Find $f''(x)$ and set $f''(x) = 0$ to find the critical values of f'. Remember also to consider the undefined cases.
2. Construct a *second derivative sign chart* and label the critical values of f'. Since f'' is continuous, it cannot change signs without passing through $y = 0$ by IVT.
3. Test a value in each interval between critical values of f' to check whether f'' is positive or negative. Label these on the sign chart.
4. Classify each critical value of f' as a point of inflection or not of f by observing the sign chart. If required, find the coordinate of the points of inflection by substituting into $f(x)$ (not $f'(x)$ or $f''(x)$!)
5. If asked to justify your answers, refer directly to the Definitions and Theorems.

Mind the wording in the question again; if we are asked to find the *location* of the points of inflection, the x-values suffice. If we are asked to find the points of inflection themselves, give the coordinate.

Many questions will require us to find *every* piece of information related to f. That is, find the intervals for which f is increasing or decreasing, locate and classify all critical values of f, find the intervals for which f is concave up or concave down, and find all points of inflection. It may sound very confusing, but the process is time consuming. Most of the computations are really quite simple, and the key is remembering the Definitions and Theorems. Most things follow directly from the related Definitions and Theorems.

Be clear with your explanations as well. It is not satisfactory to say "f has a local minimum at $x = 4$ since *it* changes from negative to positive at $x = 4$" or "f has an inflection point at $x = -3$ because *it* changes sign there." It is much more accurate to say, for example, "f has a local minimum at $x = 4$ since f' changes from negative to positive at $x = 4$."

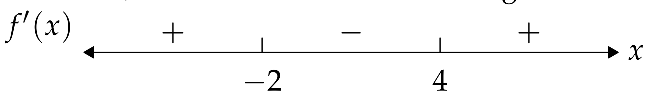
Examples 3.17. For the following examples, find the intervals for which the given function is increasing or decreasing, locate and classify all critical values, find the intervals for which the function is concave up or concave down, and find all points of inflection. Justify your answer in every case.

1. $f(x) = x^3 - 3x^2 - 24x + 1$

   First, we find the critical values of f:

   $$f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x+2)(x-4) = 0 \Longrightarrow x = -2, 4$$

   From this, we make a first derivative sign chart and fill it in:

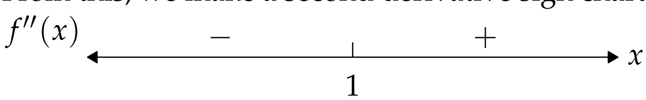


   $f'(-3) = 3(-)(-) = +$
   $f'(0) = 3(+)(-) = -$
   $f'(5) = 3(+)(+) = +$

   Now we can also find the critical values of f':

   $$f''(x) = 6x - 6 = 0 \Longrightarrow x = 1$$

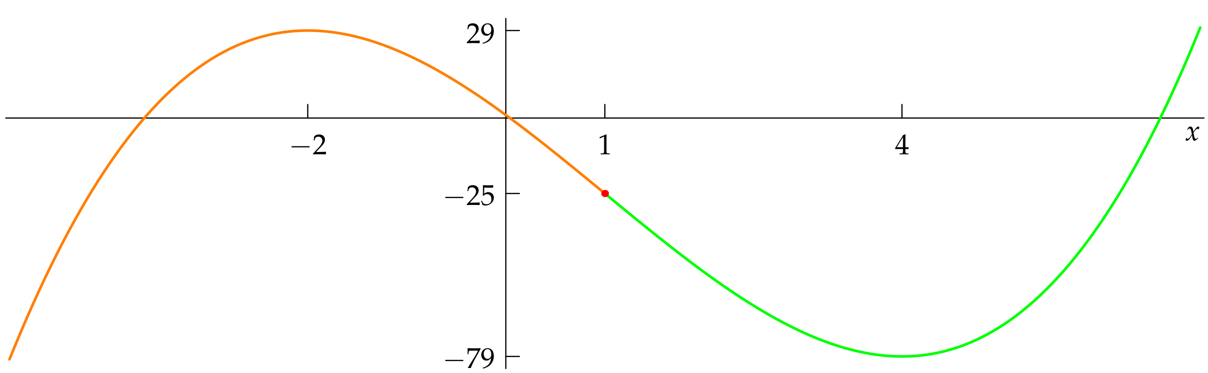
   From this, we make a second derivative sign chart and fill it in:



   $f''(0) = -6 = - < 0$
   $f''(2) = 6 = + > 0$

   The derivative charts allow us to conclude the following:

   f is increasing on the intervals $(-\infty, -2)$ and $(4, \infty)$, since $f'(x) > 0$.
   f is decreasing on the interval $(-2, 4)$, since $f'(x) < 0$.
   f has a local maximum at $x = -2$ or point $(-2, 29)$, since f' changes from positive to negative.
   f has a local minimum at $x = 4$ or point $(4, -79)$, since f' changes from negative to positive.
   f is concave up on the interval $(1, \infty)$, since $f''(x) > 0$.
   f is concave down on the interval $(-\infty, 1)$, since $f''(x) < 0$.
   f has a point of inflection at $x = 1$ with coordinate $(1, 25)$, since f'' changes sign.



2. $g(x) = x + 1 + \frac{1}{x}$

First, we find the critical values of g:

$$g'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2} = 0 \Longrightarrow x = -1, 0, 1 \qquad \text{(undefined at } x = 0\text{!)}$$

from which we can make a first derivative sign chart:

$g'(x)$: $+$ | -1 | $-$ | 0 | $-$ | 1 | $+$ → x

$g'(-2) = 1 - \frac{1}{4} = \frac{3}{4} > 0$
$g'(-\frac{1}{2}) = 1 - 4 = -3 < 0$
$g'(\frac{1}{2}) = 1 - 4 = -3 < 0$
$g'(2) = 1 - \frac{1}{4} = \frac{3}{4} > 0$

Now we can find the critical values of g':

$$g''(x) = 2x^{-3} = \frac{2}{x^3} = 0 \Longrightarrow x = 0 \qquad \text{(only undefined at } x = 0\text{)}$$

The second derivative sign chart is:

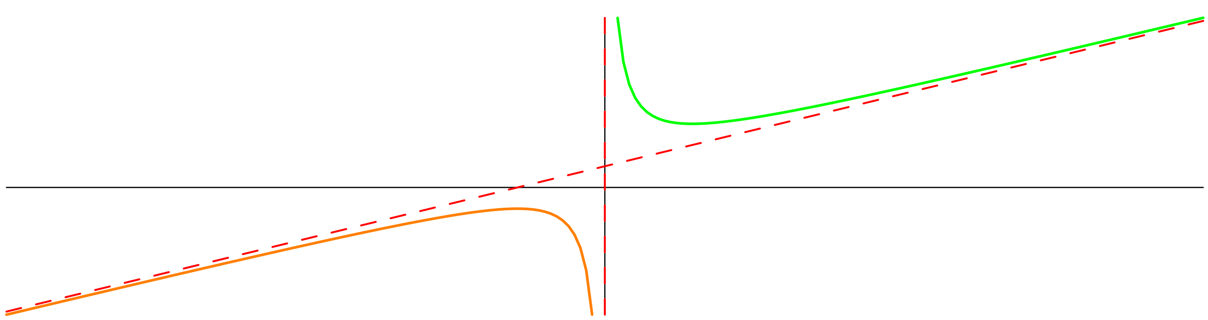
$g''(x)$: $-$ | 0 | $+$ → x

$g''(-1) = \frac{2}{-1} = -2 < 0$
$g''(1) = \frac{2}{1} = 2 > 0$

From the sign charts, we can conclude the following:

g is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$, since $g'(x) > 0$.
g is increasing on the intervals $(-1, 0)$ and $(0, 1)$, since $g'(x) < 0$.
g has a local maximum at $x = -1$ or point $(-1, -1)$, since g' changes from positive to negative.
g has a local minimum at $x = 1$ or point $(1, 3)$, since g' changes from negative to positive.
g is concave up on the interval $(0, \infty)$, since $g''(x) > 0$.
g is concave down on the interval $(-\infty, 0)$, since $g''(x) < 0$.
g does *not* have a point of inflection; even though g'' changes sign at $x = 0$, $g(0)$ is not defined!

Note that the g has a vertical asymptote at $x = 0$ and a *slant asymptote* of $y = x + 1$. Don't worry if you can't remember slant asymptotes!

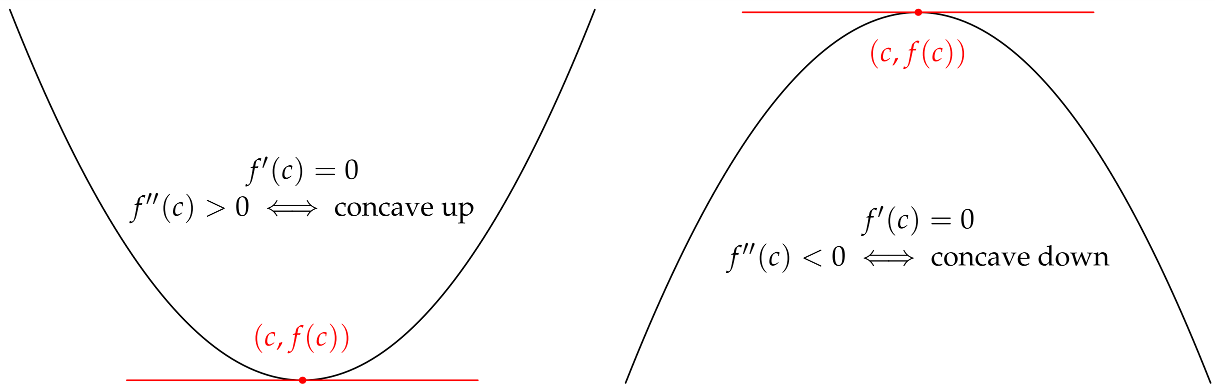


Recall that the first derivative test allows us to classify critical points as local maxima, local minima, or neither. We can also do something similar using the *second derivative*.

Theorem 3.18 (Second Derivative Test). Suppose f is continuous on an interval I containing $x = c$.

If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at $x = c$.

If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at $x = c$.



Hopefully the second derivative test is visually clear; since $f'(c) = 0$, $x = c$ is a critical value of f, so $f(c)$ must be a local minimum, local maximum, or neither. If the curve is concave up at that point, we must have a minimum, and if the curve is concave down at that point, we must have a maximum. If $f'(c) = 0$ and $f''(c) = 0$, then f could have either a local minimum, a local maximum, or a point of inflection at $x = c$: the test is *inconclusive*. Use the first derivative test if the second derivative test does not produce a suitable conclusion.

Example 3.19. Show that $g(x) = 5x^2 - 10x$ has a critical value at $x = 1$, and identify whether it is a local minimum or local maximum using the second derivative test.

$$g'(x) = 10x - 10 = 0 \Longrightarrow x = 1 \text{ is a critical value.}$$
$$g''(x) = 10 > 0 \Longrightarrow g \text{ is concave up for all } x.$$

Since $g'(1) = 0$ and $g''(1) = 10 > 0$, g has a relative minimum at $x = 1$ by the second derivative test.

Derivative Relationships

We now know how the sign of f' or f'' affect the behavior of f, but there are situations in which we are not given an equation for $f(x)$. Instead, we may be given the *graph* of f'; we need to know how information about f' can help us in other ways.

Theorem 3.20. Suppose f is differentiable on an interval I containing $x = c$. Then:

f is concave up on $I \iff f'$ is increasing on I

f is concave down on $I \iff f'$ is decreasing on I

f has a point of inflection at $x = c \iff f'$ has a local minimum or local maximum at $x = c$

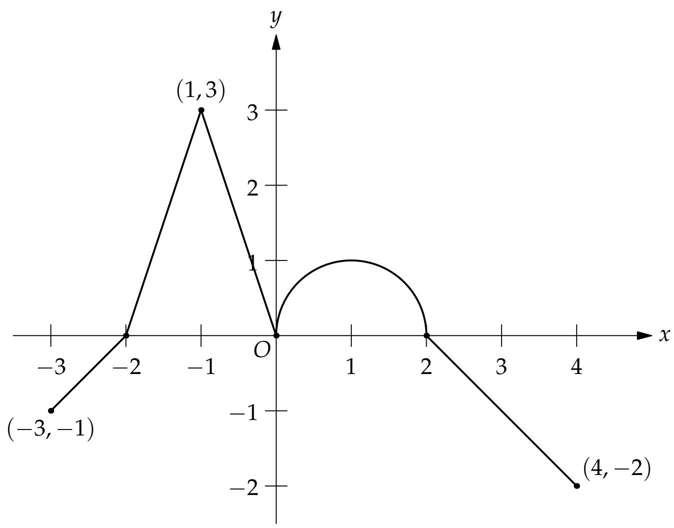
The key for understanding this is realizing that f' is its own function, and its first derivative is f''. So all of the statements in Theorem 3.8 and 3.10 apply to f'. That is,

f' increasing $\iff$ $f''(x) > 0$ $\iff$ f is concave up

f' decreasing $\iff$ $f''(x) < 0$ $\iff$ f is concave down

f' has a local extreme $\iff$ $f''(x)$ changes sign $\iff$ f has a point of inflection

Examples 3.21. Each of the following examples is similar to what you would expect on an AP free-response question. Be clear in your explanations!



Graph of f'

1. Let f be a differentiable function. On the interval $-3 \leq x \leq 4$, the graph of f', the derivative of f, consists of four line segments and a semicircle, as shown in the figure above.

   (a) Find all intervals for which f is decreasing. Justify your answer.

   (b) Find all intervals for which the graph of f is concave up. Explain your reasoning.

   (c) Does f have a relative minimum, relative maximum, or neither at $x = 0$? Justify your answer.

   (d) Give the x-coordinates of the points of inflection of the graph of f for $-3 < x < 4$. Justify your answer.

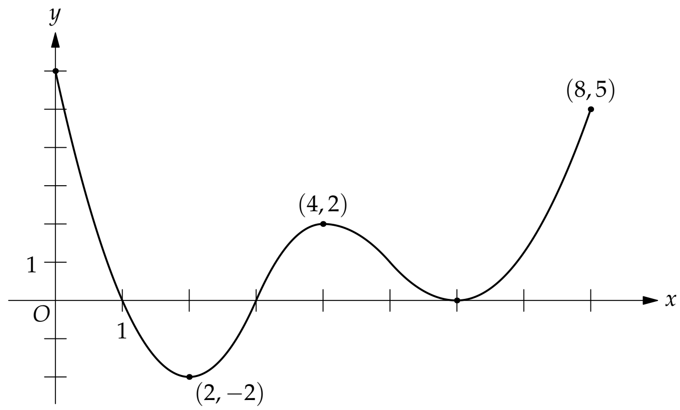
   Sample response:

   (a) f is decreasing on the intervals $(-3, -2)$ and $(2, 4)$ because $f'(x) < 0$ on these intervals.

   (b) The graph of f is concave up on the intervals $(-3, -2)$, $(-2, 1)$ and $(0, 1)$ since f' is increasing on these intervals.

   (c) $f'(x) > 0$ for $(-2, 0)$ and $f'(x) < 0$ for $(0, 2)$. $f'(x)$ does not change sign at $x = 0$, so f has neither a relative minimum nor a relative maximum at this location.

   (d) The graph of f has a point of inflection at each of $x = -1$, $x = 0$, and $x = 1$ because $f'(x)$ changes from increasing to decreasing at $x = -1$ and $x = 1$, and from decreasing to increasing at $x = 0$.



Graph of g'

2. Let g be a twice-differentiable function defined on the interval $[0, 8]$ which satisfies $g(2) = 5$. The graph of g', the derivative of g, is shown in the figure above.

   (a) On what open intervals is the graph of g both increasing and concave down? Give a reason for your answer.

   (b) Let h be the function defined by $h(x) = g(x) \cdot g'(x)$. Find $h'(2)$.

   (c) Find the equation of the tangent line to the curve g at $x = 2$.

   (d) Find the average rate of change of g' on the interval $[1, 6]$. Does the Mean Value Theorem guarantee a value c, $1 < c < 6$, for which $g''(c)$ is equal to this average rate of change? If so, find any such values c. Justify your answer.

Sample response:

(a) The graph of g is both increasing and concave down on intervals $(0, 1)$ and $(4, 6)$ since $g'(x)$ is both positive and decreasing on these intervals.

(b) We can use the product rule:

$$h'(x) = g'(x) \cdot g'(x) + g(x) \cdot g''(x)$$
$$\Longrightarrow h'(2) = \left[g'(2)\right]^2 + g(2) \cdot g''(2) = (-2)^2 + (5)(0) = 4$$

whence $g''(2) = 0$ is the slope of the tangent to g' at $x = 2$.

(c) We have $g(2) = 5$ and $g'(2) = -2$, so the tangent line equation at $x = 2$ is

$$y - g(2) = g'(2)(x - 2) \Longrightarrow y - 5 = -2(x - 2)$$

(d) The average rate of change of g' on $[1, 6]$ is

$$\frac{g'(6) - g'(1)}{6 - 1} = \frac{0 - 0}{5} = 0$$

Since g is twice-differentiable on $[0, 8]$, g' is continuous and differentiable on the same interval. The hypotheses of MVT are satisfied, and thus there exists a value $x = c$ such that $g''(c) = 0$. By inspection, we have two such values $c_1 = 2$ and $c_2 = 4$.

Exercises 3.2. 1. For each of the following functions, find the intervals for which the given function is increasing or decreasing, locate and classify all critical values, find the intervals for which the function is concave up or concave down, and find all points of inflection. Justify your answer in every case.

(a) $f(x) = \ln\left(x^2 + 3\right)$

(b) $g(x) = \csc \frac{x}{2}$ for $-2\pi \leq x \leq 2\pi$

(c) $h(x) = x^3 - 3x^2 - 9x + 2$

(d) $y = \frac{1}{2}\left(e^x + e^{-x}\right)$

2. An important function in statistics is the *standard normal distribution function*[16], given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

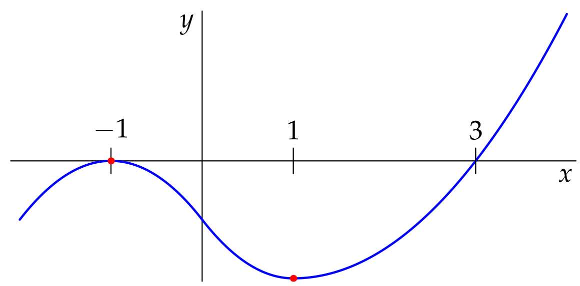
(a) Find the critical points of f and find the intervals where f is increasing or decreasing.

(b) Find all points of inflection.

(c) Find $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$. Discuss the *horizontal asymptotes* of f.

3. Let f be a twice-differentiable function. The graph of $y = f'(x)$ is shown on the right for the interval $-2 \leq x \leq 4$.

f' has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.



(a) Identify all critical values of f.

(b) Give the location of all points of inflection of f.

(c) To the best of your ability, sketch what the graph of f could look like.

(d) Would the second derivative test be applicable to f at $x = -1$? Why or why not?

4. If $f(x) = \arctan x + \operatorname{arccot} x$, find $f'(x)$ and $f''(x)$. What can we conclude about $f(x)$?

5. Let g be a twice-differentiable function satisfying $g(3) = 4$, $g'(3) = 0$, and $g''(3) = -2$. Does g have a local minimum, local maximum, or neither at $x = 3$?

6. (Hard) The standard form of a cubic polynomial is $y = ax^3 + bx^2 + cx + d$, for constants a, b, c, d and $a \neq 0$.

(a) Show that a cubic polynomial has at most two local extrema and exactly one point of inflection.

(b) Find a formula for the locations of the local extrema and the points of inflection in terms of the constants a, b, c, d. Under what conditions does the cubic have zero, one, or two local extrema?

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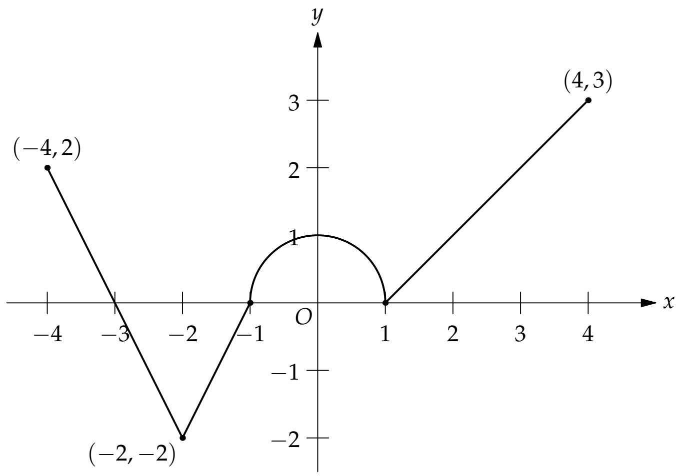
[16]For those who have studied statistics, the graph of this function is the *bell-shaped curve*.



Graph of h'

7. Let h be a differentiable function defined on the interval $[-4, 4]$. The graph of h', the derivative of h, consists of three line segments and a semicircle centered at the origin, as shown in the figure above.

   (a) Does h have a relative minimum, a relative maximum, or neither at $x = -1$? Give a reason for your answer.

   (b) Give the x-coordinates of the points of inflection of the graph of h for $-4 < x < 4$. Justify your answer.

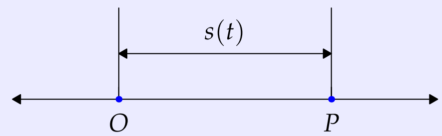
   (c) Let H be the function defined by $H(x) = x - h(x)$. On what intervals, if any, is H decreasing for $-4 \le x \le 4$? Show the analysis that leads to your answer.

   (d) Find the values of each $h''(-3)$ and $h''(1)$, or show that it does not exist. Explain your reasoning.

3.3 Motion of an Object and Other Real-World Applications

Recall that the field of calculus was developed in large part by Sir Isaac Newton, who you probably know was a physicist in his time. Unsurprisingly, many of Newton's early contributions to calculus were purposed to answer questions in physics. Here, we consider the simplest possible application of calculus to mechanics.

Definition 3.22 (Position). Suppose an object P moves along a straight line so that its position s from an origin O is given as some function of time t. We write $s = s(t)$ where $t \geq 0$, and $s(t)$ is called the *position function* or *displacement function* of P.



The sign of $s(t)$ also indicates the direction from O. Another common notation for the position function of an object is $x(t)$.

Theorem 3.23. On the horizontal axis through the origin O:

$s(t) > 0 \iff P$ is located to the *right* of O.

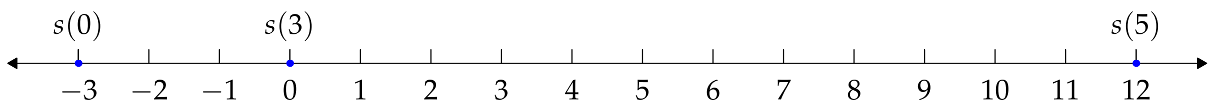
$s(t) < 0 \iff P$ is located to the *left* of O.

$s(t) = 0 \iff P$ is located *at* O.

Example 3.24. Suppose a particle moves along a line such that its position at time t is

$$s(t) = t^2 - 2t - 3 \quad \text{for } t \geq 0$$

The particle's *initial position* is $s(0) = -3$; it lies 3 units to the left of the origin at $t = 0$.
Its position at $t = 3$ is $s(3) = 0$, so it lies at the origin.
At $t = 5$, the particle's displacement is $s(5) = 12$, so it lies 12 units to the right of the origin.



Be sure you can visualize the motion of an object described here; the object in question *only* moves in a straight line. Of course, in calculus, we should also consider the *rate* at which the particle moves.

Definition 3.25 (Velocity). The *average velocity* of an object moving in a straight line in the time interval from $t = a$ to $t = b$ is the change in position divided by the change in time. If $s(t)$ is the position function for an object, then

$$\text{average velocity} = \frac{s(b) - s(a)}{b - a}$$

The *instantaneous velocity* of the object at time t or simply *velocity function* $v(t)$ is the derivative of $s(t)$:

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

As we should know, on a graph of $s(t)$ the instantaneous velocity at a particular time is the slope of the tangent to the graph at that point.

The idea of velocity should be fairly intuitive. No doubt students know rate $= \frac{\text{distance}}{\text{time}}$. If the position function of an object $s(t)$ measures distance, then its rate of change with respect to time is precisely the velocity $v(t)$.

Examples 3.26. 1. Suppose John begins driving starting 31 km away from his house on a straight road at 8 a.m. in the opposite direction of his house. At 12 p.m., he is 478 km away from his house. If we let the function $s(t)$ represent John's position relative to his house measured in kilometers and t be measured in hours since 8 a.m., then John's average velocity over the time interval $0 \le t \le 4$ is

$$\text{average velocity} = \frac{s(4) - s(0)}{4 - 0} = \frac{478 - 31}{4 - 0} = 111.75\,\text{km/h}$$

Note the units for velocity km/h. Indeed it is a rate with respect to time!

2. We return to Example 3.24. If a particle's position is given by $s(t) = t^2 - 2t - 3$, then its instantaneous velocity at any time t is given by

$$s'(t) = v(t) = 2t - 2$$

The particle's *initial velocity* is $v(0) = -2$. That is, the particle's position is *decreasing* at a rate of 2 units distance per unit time t. Otherwise said, the particle is moving *left*.

At $t = 1$, the particle's velocity is $v(1) = 0$; the particle is *not moving* at this moment.

And at $t = 3$, the particle's velocity is $v(3) = 4$. It is moving *right*.

We will formalize the language moving right, moving left, etc. later.

It is also worth discussing the *second derivative* of position $s(t)$.

Definition 3.27 (Acceleration). The *average acceleration* of an object moving in a straight line in the time interval from $t = a$ to $t = b$ is the change in velocity divided by the change in time. If $v(t)$ is the position function for an object, then

$$\text{average acceleration} = \frac{v(b) - v(a)}{b - a}$$

The *instantaneous acceleration* of the object at time t or *acceleration function* $a(t)$ is the derivative of $v(t)$, which is in turn the second derivative of $s(t)$:

$$a(t) = v'(t) = s''(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

Usually it is quite difficult to interpret the meaning of a second derivative in any context when learning, but in this form, it should be slightly more digestible. Anyone who has ridden in a car as the idea that when the vehicle accelerates, the velocity of the car is *increasing*. And when the vehicle decelerates, i.e. the acceleration is negative, then the velocity of the car is *decreasing*.

The units for the values of $s(t)$, $v(t)$, and $a(t)$ are consistent with what they represent. Each time we differentiate with respect to time t, we calculate a rate per unit of time. So, depending on the units of $s(t)$ and t:

the units of position $s(t)$ could be m, ft, in, cm, etc.

the units of velocity $v(t)$ could be m/s, ft/min, in/yr, cm/h, etc.

the units of acceleration $a(t)$ could be $\mathrm{m/s^2}$, $\mathrm{ft/min^2}$, $\mathrm{in/yr^2}$, $\mathrm{cm/h^2}$, etc.

where, for example, $\mathrm{m/s^2}$ is 'meters per second per second'.

Example 3.28. A particle moves in a straight line with displacement from the origin given by function $s(t) = 4t - 3t^2$ inches at time t seconds. Then:

The position of the particle at $t = 3$ is $s(3) = -15$ in, or 15 in left of the origin.

$v(t) = 4 - 6t$, and the velocity of the particle at $t = 3$ seconds is $v(3) = -14$ in/s

$a(t) = -6$ is constant, so the acceleration of the particle at $t = 3$ seconds is $a(3) = -6\,\mathrm{in/s^2}$

We should realize by now that everything that has been discussed in this section is simply a rephrasing of the things we learned in Sections 3.1 and 3.2. By letting $f = s$, $f' = v$, and $f'' = a$, every previously mentioned Definition and Theorem applies to these motion functions!

Theorem 3.29. On the horizontal axis through the origin O, if $s(t)$ represents the position of an object P relative to O and $v(t)$ its velocity:

$v(t) > 0 \iff s(t)$ is increasing $\iff P$ is moving right

$v(t) < 0 \iff s(t)$ is decreasing $\iff P$ is moving left

$v(t) = 0 \iff s(t)$ is constant or has a local extreme $\iff P$ is at rest

And if $a(t)$ represents the acceleration of object P:

$a(t) > 0 \iff v(t)$ is increasing $\iff$ the graph of $s(t)$ is concave up

$a(t) < 0 \iff v(t)$ is decreasing $\iff$ the graph of $s(t)$ is concave down

$a(t) = 0 \iff v(t)$ is constant or has a local extreme

Let $s(t)$ be differentiable and $v(c) = 0$ for some time $t = c$. Then one of the following must occur:

If $v(t)$ changes from positive to negative at $t = c$, then P changes direction from right to left.

If $v(t)$ changes from negative to positive at $t = c$, then P changes direction from left to right.

If $v(t)$ does not change sign at $t = c$, then P does not reverse direction.

The reasoning should be obvious if you've spent enough time studying relationships between a function and its derivatives. The methods we used in the previous sections for finding intervals for which f is increasing, decreasing, finding relative extrema, etc. are also useful here!

Example 3.30. A particle moves in a straight line according to the function $s(t) = t^3 - 6t^2 + 9t$ cm, where t is the time in seconds for $t \geq 0$. Find expressions for the particle's velocity and acceleration, then find the particle's initial position, velocity, and acceleration. Also give every time interval for which the particle is moving left or right, and find all times for which the particle reverses direction.

First, we are given

$$s(t) = t^3 - 6t^2 + 9t \implies v(t) = 3t^2 - 12t + 9 \implies a(t) = 6t - 12$$

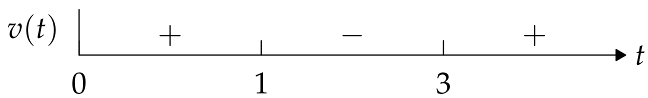
For which the initial values of the particle's position, velocity, and acceleration are:

$$s(0) = 0\,\text{cm}, \quad v(0) = 9\,\text{cm/s}, \quad a(0) = -12\,\text{cm/s}^2$$

To find the desired time intervals, we need the critical values of $s(t)$:

$$v(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3) = 0 \implies t = 1, 3\,\text{s}$$

Now we can construct a first derivative chart for velocity:



Note that since $s(t)$, and in turn $v(t)$, is only defined for $t \geq 0$, we need only consider positive values of t for our chart.

The particle is moving to the right for $0 \leq t < 1$ and $t > 3$ seconds.
The particle is moving to the left for $1 < t < 3$ seconds
The particle reverses direction at $t = 1$ s and $t = 3$ s since $v(t)$ changes sign at those times.

Again, all these methods are identical to what we've already done, up to a few notation changes.

Speed

Some students may have heard in a science class for example that velocity is a quantity which measures both speed and direction. This is consistent with our definitions: the speed of an object P is the numerical value of $v(t)$, and its direction is represented by the sign of $v(t)$. For example, if a particle has $v(2) = -4$, its speed at $t = 2$ is 4, and its direction is left, as indicated by the negative sign.

Definition 3.31 (Speed). The *speed* of an object P at any time t is the absolute value[17] of its velocity at that time, $|v(t)|$.

Taking absolute value eliminates the direction component of velocity. So speed simply measures *how fast* something is traveling, regardless of the direction of travel.

Again, this should be mostly intuitive. Suppose you are driving a car forward on a straight road at 20 mi/h ($v(t) = 20$). At a later time, you are driving backward (reversing) on the road at 20 mi/h ($v(t) = -20$). In either case, you are traveling at a speed of 20 mi/h.

[17] Since velocity is a *vector quantity*, it is more accurate to say that speed is the *magnitude* of the object's velocity. However, we don't need this definition until we study movement of objects in higher dimensions, which we will do in BC.

To determine when the speed of an object P with position $s(t)$ is increasing or decreasing, we need to employ the following Theorem.

Theorem 3.32. Let the function $s(t)$ represent the position of an object P moving on a line at time t.

If the signs of velocity $v(t)$ and acceleration $a(t)$ are the same (both positive or both negative), i.e. $v(t) \cdot a(t) > 0$, then the speed of P is increasing (P is *speeding up*).

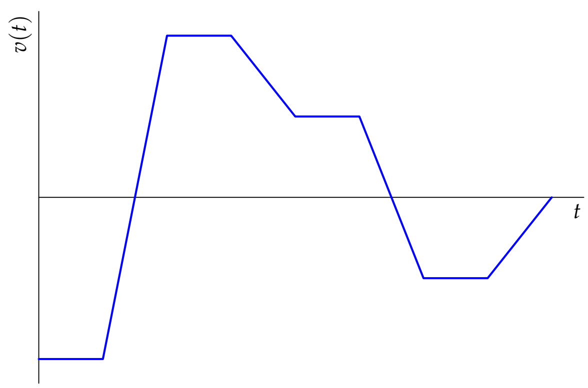
If the signs of $v(t)$ and $a(t)$ are opposite, i.e. $v(t) \cdot a(t) < 0$, then the speed of P is decreasing (P is *slowing down*).

If one of $v(t)$ or $a(t)$ is zero, i.e. $v(t) \cdot a(t) = 0$, then P is neither speeding up nor slowing down and we may refer to Theorem 3.29.

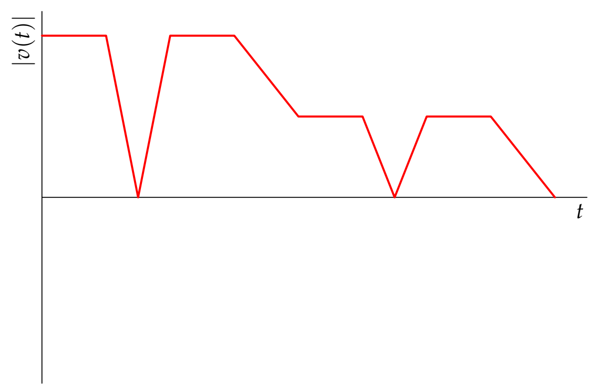
Use your intuition here: Suppose $v(t)$ is positive. If $a(t)$ is positive, then $v(t)$ is increasing, and thus the numerical value (the speed) of $v(t)$ is going up.

Now suppose $v(t)$ is negative. If $a(t)$ is also negative, then $v(t)$ is decreasing, and thus the numerical value of $v(t)$ is going down, or becoming *more negative*. So its absolute value is *increasing*.

Another way we can think about it is to recall how absolute value transforms the graph of a function. $|f(x)|$ takes any negative values of $f(x)$ and makes them positive. Visually, we take the portions of the graph of f below the x-axis and reflect them about the x-axis, and leave the portions of the graph above or on the x-axis untouched.



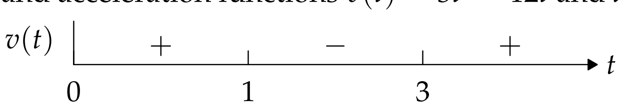
Graph of $v(t)$

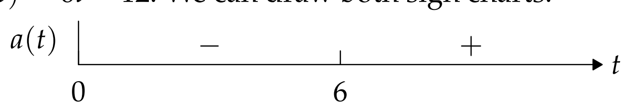


Graph of $|v(t)|$

When $v(t) < 0$ (is below the t-axis) but is increasing on some interval, the absolute value transformation reflects it so that speed $|v(t)|$ is *decreasing* on that interval. A similar argument could be made for every other case.

Example 3.30 cont. For the position function $s(t) = t^3 - 6t^2 + 9t$, we had the corresponding velocity and acceleration functions $v(t) = 3t^2 - 12t$ and $a(t) = 6t - 12$. We can draw both sign charts:





We can say that the particle is speeding up on time intervals $(1, 3)$ and $(6, \infty)$ since $v(t)$ and $a(t)$ have the same sign on those intervals, and slowing down on $(0, 1)$ and $(3, 6)$ because $v(t) \cdot a(t) < 0$ on those intervals.

Other Rates of Change

We have seen that if $s(t)$ models the position of an object, then $s'(t) = v(t)$ is the rate of change in position with respect to time, which we call velocity.

However, there are many more real-world examples where we can use rates of change! For example,

- temperature varies continuously
- a species' population changes with respect to time
- the volume of water in a tank can be modeled using functions

The key is to be wary of *units*. Each time we differentiate a given function with respect to variable □, we must divide the unit of the function value by the unit of □. In most applications, we will be differentiating with respect to time.

Examples 3.33. 1. (Calculator) Let $S(t) = 6.2t^4e^{-0.5t}$ represent the concentration of medication in the bloodstream, where t is measured in hours and $S(t)$ measured in nanograms per millimeter. Find $S'(10)$ and interpret the result using correct units.

We can use our graphing calculator to find $S'(10) \approx -43.123$. At $t = 10$ hours, the concentration of medication in the bloodstream is decreasing at a rate of 43.123 nanograms per millimeter per hour (or $43.123\,\text{ng/mm}\cdot\text{h}=43.123\,\text{ng}\cdot\text{mm}^{-1}\cdot\text{h}^{-1}$).

Note that since we said 'decreasing at a rate of...', it is redundant to put a negative sign in front of the rate in our explanation. And since the unit for t is hours, when we differentiate, we add a 'per hour' to the unit.

2. (Calculator) Suppose a manufacturer produces hats. The cost in dollars of producing x hats in a factory each day is given by

$$C(x) = 0.00017x^3 + 0.003x^2 + 4x + 2300$$

Find $C(200)$ and $C'(200)$ and interpret these results using correct units.

Our calculator gives $C(180) \approx 4108.64$ and $C'(180) \approx 21.604$. We can interpret as such:

- The cost of producing 180 hats in a factory each day is $4108.64.
- The rate at which costs are increasing with respect to the number of hats produced when 150 hats are made per day is 21.604 dollars per hat.

3. (Calculator) The height in centimeters of a species of raspberry bush grown in California t years after it is planted is given by

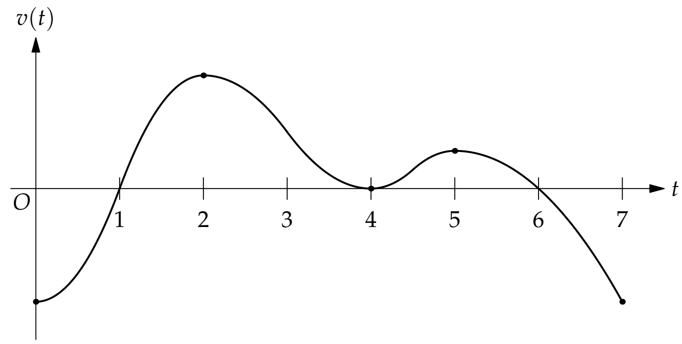
$$H(t) = 17\ln(1.85t + 2) + 29, \quad t \geq 0$$

Find $H''(8)$ and interpret this value using correct units.

With a calculator, we get $H''(8) \approx -0.206$. At $t = 8$ years after the bush was planted, the rate at which the height of the raspberry bush is changing is decreasing by $0.206\,\text{cm/yr}^2$.

Interpreting the value of a second derivative (the rate of a rate) is often challenging, but having correct units is a good place to start.

Exercises 3.3. For the first question, use the graph shown below.



Graph of v

1. A particle moves along the x-axis so that its velocity at time t, for $0 \leq t \leq 7$, is given by a differentiable function v whose graph is shown above. The graph has horizontal tangents at $t = 2$, $t = 4$, and $t = 5$.
   (a) Find all time intervals where the particle is moving to the right. Justify your answer.
   (b) For how many values of t does the particle reverse direction? Justify your answer.
   (c) On the interval $2 < t < 4$, is the speed of the particle increasing or decreasing? Explain your reasoning.
   (d) During what time intervals, if any, is the acceleration of the particle positive? Explain your reasoning.

2. (Calculator) A stone is launched vertically so that its position, in feet, above ground level after t seconds is given by

   $$s(t) = 321.5t - 16.1t^2, \quad t \geq 0$$

   (a) Find the velocity and acceleration functions for the stone.
   (b) Find the maximum height achieved by the stone.
   (c) What is the time taken for the stone to hit the ground?
   (d) Find $s'(3)$ and interpret this value in context using correct units.

3. A particle moves along a horizontal line with displacement relative to O given by

   $$x(t) = 1 - 2\sin t, \quad 0 \leq t \leq 2\pi$$

   where t is measured in minutes and $x(t)$ is measured in centimeters.
   (a) Find the initial position, velocity, and acceleration of the particle.
   (b) During what time intervals is the particle's speed decreasing?
   (c) At what time t for $0 \leq t \leq 2\pi$ is the particle furthest to the right? Justify your answer. (*Hint: Furthest to the right means absolute maximum position!*)
   (d) Find $x''(\frac{\pi}{4})$. Using correct units, interpret this value in context.

4. Water is draining from an underground reservoir. The remaining volume of water, measured in cubic meters, t hours after 1 p.m. is given by

$$V = 180(30 - t)^2$$

   (a) Find the average rate at which water is leaving the reservoir in the first 5 hours.
   (b) Find the instantaneous rate at which water is leaving at 6 p.m.
   (c) At what time will the reservoir be empty?

5. A particle's position along a horizontal axis relative to the origin is described by the twice-differentiable function s, where $s(t)$ is measured in inches and t is measured in seconds. Selected values of $s(t)$ are shown in the table below.

t (seconds)	0	1	3	5	6	9	11
$s(t)$ (inches)	4	3	-1	4	5	2	-7

   (a) Approximate $v(2)$ and provide your answer with correct units.
   (*Hint: use the average velocity from* $t = 1$ *to* $t = 3$.)
   (b) Explain why on the time interval $(0, 5)$, there must be a time t where the particle is at rest.
   (c) What is the minimum number of times the particle must be at the origin? Justify your answer.

6. (Calculator) When a glasses manufacturer produces and sells x pairs of glasses per day, their revenue, in dollars, is given by

$$R(x) = 170 \ln\left(1 + \frac{x}{90}\right) + 800$$

The cost to manufacture x pairs of glasses per day, in dollars, is given by

$$C(x) = (x - 70)^2 + 250$$

   (a) Write a function $P(x)$ that models the company's *profit* by producing and selling x pairs of glasses per day.
   (b) From your answer in part (a), find $P'(50)$ and interpret this value using correct units.

3.4 Optimization Problems

Many real-life problems can be rephrased in terms of maximizing or minimizing the value of a function. For instance, 'How do we make the most profit?' (maximize income, or minimize cost). Calculus has a role to play in answering these questions. This is the very definition of *optimization* in calculus.

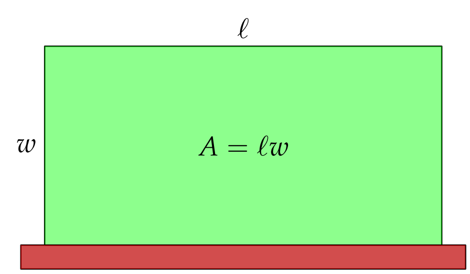
In essence, our method for optimization problems is a simple application of the things we learned in the previous sections!

1. If it is possible or helpful, draw a picture that models the given situation. This goes for many word problems in mathematics!
2. Convert the word problem into the form 'Find the maximum/minimum value of a function.' This is typically the difficult part, since the word problem may not provide explicit equations or variables, so you may need to write your own. Usually there will be two equations, where one is the *constraint*: you will need to isolate a variable in the constraint and make a substitution later.
3. Differentiate the function and find its extrema using the methods we discussed in the previous sections.
4. Find the desired answer using correct units and interpret it. Some solutions may not be physically relevant.

Follow these steps and do many examples to nail optimization problems; mastery comes with extensive practice!

Examples 3.34. 1. A three-sided rectangular fence is to be built against a wall to make a field. Find the dimensions of the field for which its area is a maximum if there is 50 m of fencing available.

First, we draw a picture (shown on the right) and introduce variables: let ℓ be the length parallel to the wall, and w be the width of the field. Note that 50 m of fencing refers to the *perimeter* of the rectangle aside from one width (because of the wall). So we can write



$$50 = \ell + 2w \implies \ell = 50 - 2w \quad \text{and}$$
$$\text{Area: } A = \ell w = (50-2w)w = 50w - 2w^2$$

The problem can now be rephrased: find the *maximum* of the function $A = 50w - 2w^2$. Differentiate to find the only critical value

$$\frac{\mathrm{d}A}{\mathrm{d}w} = 50 - 4w = 0 \implies w = 12.5$$

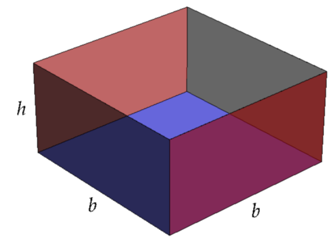
Either observe that $y = A(w)$ is a downward opening parabola or use the first or second derivative test to see that A has a maximum at $w = 12.5\,\mathrm{m}$, from which we can find ℓ:

$$\ell = 50 - 2(12.5) = 25\,\mathrm{m}$$

The dimensions of the fenced field that produces a maximum area is therefore $25\,\mathrm{m} \times 12.5\,\mathrm{m}$ and the maximum area is $A = 25\,\mathrm{m} \cdot 12.5\,\mathrm{m} = 312.5\,\mathrm{m}^2$ (though the area wasn't asked for).

2. A box with volume $4\,\text{ft}^3$ is to be made with a square base and no lid. Find the dimensions of the box that produces a minimal surface area.

We can draw a picture and label each edge of the box. Since the base is square, we can label two adjacent edges b and the remaining edge h. We have a volume constraint, so we can write



$$V = 4 = b^2 h \Longrightarrow h = \frac{4}{b^2} \quad \text{and}$$

$$\text{Surface Area: } A = b^2 + 4bh = b^2 + 4b\left(\frac{4}{b^2}\right)$$

$$= b^2 + 16b^{-1}$$

The problem can be rephrased as minimizing the function $A = b^2 + 16b^{-1}$. We differentiate:

$$A'(b) = 2b - 16b^{-2} = \frac{2(b^3 - 8)}{b^2} = 0 \Longrightarrow b = 0, 2$$

In the context of this problem, $b = 0$ is not physically possible, so we need only test $b = 2$. If we used the first or second derivative test, we would see that $A(2)$ is indeed a local minimum. Then we can find

$$h = \frac{4}{b^2} = \frac{4}{2^2} = 1\,\text{ft}$$

So the dimensions of the required box are $b = 2\,\text{ft}$ and $h = 1\,\text{ft}$.

3. The sum of a number and three times another is 42. Find the maximum product of these two numbers.

In this example, it is not required to draw a picture. We can instead just define our variables and write our equations. Let x be the first number and y be the second. Then we have

$$x + 3y = 42 \Longrightarrow x = 42 - 3y$$
$$P = xy = y(42 - 3y) = 42y - 3y^2$$

whence we need to find the maximum of the function $P = 42y - 3y^2$:

$$\frac{\mathrm{d}P}{\mathrm{d}y} = 42 - 6y = 0 \Longrightarrow y = 7$$

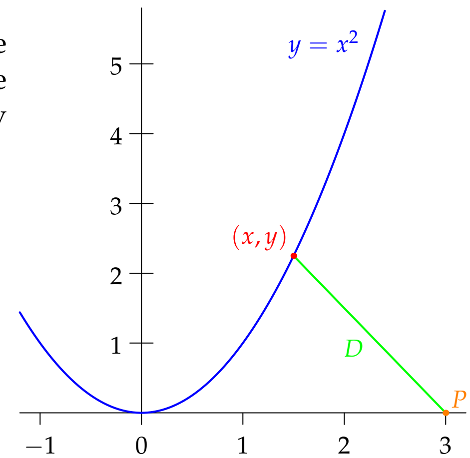
By using a derivative test, we can determine that $P(7)$ is indeed a relative maximum. Then we have

$$x = 42 - 3y = 42 - 3(7) = 21$$

So the maximum product of these two numbers is $21 \cdot 7 = 147$. Alternatively, we could have just evaluated

$$P(7) = 42(7) - 3(7)^2 = 147$$

4. Find the point on the curve $y = x^2$ which is closest to the point $P = (3, 0)$.



First, we'll draw a picture to model the question. We realize that finding the point on the parabola closest to P can be rephrased as minimizing *distance* to P. We can also simply use the equation $y = x^2$ of the curve as our constraint!

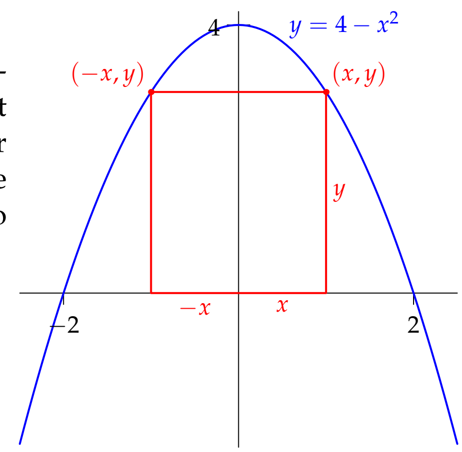
$$y = x^2$$
$$D = \sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(x - 3)^2 + (y - 0)^2}$$
$$= \sqrt{(x - 3)^2 + (x^2 - 0)^2} = \sqrt{x^2 - 6x + 9 + x^4}$$

Now differentiate:

$$D'(x) = \frac{1}{2}(x^4 + x^2 - 6x + 9)^{-1/2} \cdot (4x^3 + 2x - 6)$$
$$= \frac{4x^3 + 2x - 6}{2\sqrt{x^4 + x^2 - 6x + 9}} = \frac{2(x - 1)(2x^2 + 2x + 3)}{2\sqrt{x^4 + x^2 - 6x + 9}} = 0 \Longrightarrow x = 1$$

If you are clever, you might realize that the square root expression in the denominator is not helpful here, and in fact we could have just let $d = (x - 3)^2 + y^2$ and minimized d instead of worrying about all the square root nonsense! Regardless, we have $x = 1 \Longrightarrow y = 1^2 = 1$ and the distance-minimizing point on the parabola is $(1, 1)$.

5. A rectangle is to be inscribed beneath the parabola described by $y = 4 - x^2$ and above the x-axis. Find the maximum area of this rectangle.



The problem is difficult to imagine without drawing a picture, so it is especially helpful here! We can label the right portion of the horizontal sides of the rectangle x and $-x$ for the left. The height will be y. Again, the constraint will be the equation of the curve $y = 4 - x^2$, and what we want to maximize is *area*.

$$y = 4 - x^2$$
$$A = 2xy = 2x(4 - x^2) = 4x - x^3$$

Then we find the maximum of $A = 4x - x^3$:

$$\frac{dA}{dx} = 4 - 3x^2 \Longrightarrow x^2 = \frac{4}{3} \Longrightarrow x = \pm\frac{2}{\sqrt{3}}$$

Clearly, $x = -\frac{2}{\sqrt{3}}$ and $x = \frac{2}{\sqrt{3}}$ refer to the same dimensions. The former produces a minimum and the latter a maximum, which can be checked using a derivative test. Of course, only the positive value is physically relevant, since we are dealing with area in this example. So

$$y = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{8}{3} \Longrightarrow \max A = xy = \frac{2}{\sqrt{3}} \cdot \frac{8}{3} = \frac{16}{3\sqrt{3}}$$

Exercises 3.4. 1. A farmer would like to construct two identical rectangular fields using a total of 200 ft of fencing. This farmer constructs the two fields adjacent to each other so that they share one side of fencing. What is the largest possible area of the fields?

2. The difference of two numbers is 40. Find the two numbers such that their product is a minimum.

3. Suppose that the box in Example 3.34.2 is made of material costing \$2 per ft^2 for the base and \$1 per ft^2 for the sides. What dimensions produce the cheapest box, and what would be the cost of the box?

4. A population of a particular species of microorganisms grown in a lab at time t, for $t \geq 0$, is given by

$$P(t) = 50 + \frac{600t}{45 + t^2}$$

where t is measured in days and $P(t)$ is measured in millions of microbes.

   (a) What is the initial population of microbes?
   (b) At what time t is the population of microorganisms a maximum? Provide an exact answer and include the correct units.
   (c) (Calculator) Using the correct units, find the maximum population of microorganisms rounded to three decimals.

5. A rectangle is to be inscribed within the ellipse with equation $4x^2 + y^2 = 4$. Find the maximum possible area of this rectangle.

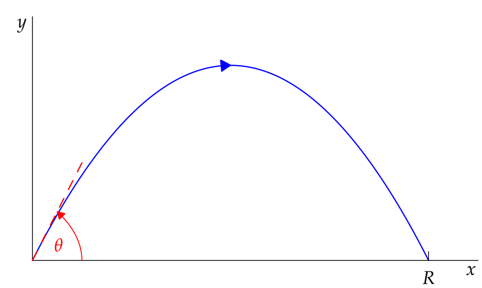
6. Find the point(s) on the curve $y = x^2$ closest to $P = (0, \frac{3}{2})$.

7. A square sheet of cardboard has side lengths of 16 inches. A box without a top is formed by trimming an equal square from each corner of the cardboard sheet and folding the remaining flaps upward. How far should we trim from each corner to form a box with the greatest possible volume?

8. An object is launched at v m/s at an angle θ radians above the horizontal. The object's path follows the parabola with equation

$$y = \frac{x}{2\cos^2\theta}\left(\sin 2\theta - \frac{g}{v^2}x\right)$$

where v, g are constant. Which angle θ gives the maximum distance R?



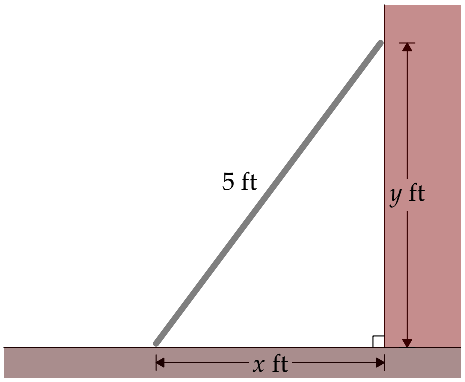
9. (Calculator) A manufacturer is designing a cylindrical container that can hold exactly 2 L or 2000 cm^3 of liquid. Find the dimensions of the container which uses the least possible amount of materials in its construction.

(*Hint: A cylinder's volume is given by* $V = \pi r^2 h$ *and surface area* $A = 2\pi rh$.)

3.5 Related Rates

In this section we consider an application of implicit differentiation. Suppose that we have an *implicit relation* involving two or more variables that are changing *with respect to time*.

Example 3.35. A 5 ft ladder is resting against a vertical wall. The ladder slips and slides down the wall, with the top of the ladder staying against the wall.

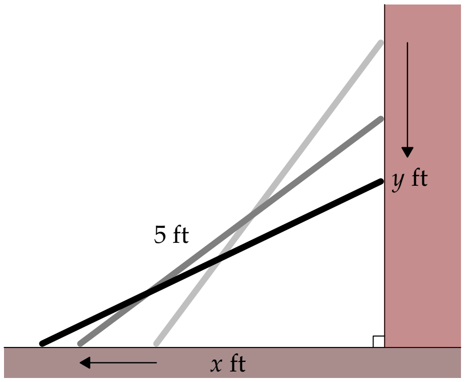


At any time, we notice that the ladder forms a right triangle with the wall and the ground. If we let x represent the distance between the wall and the bottom of the ladder and y the distance between the ground and the top of the ladder, we can use the Pythagorean Theorem to write

$$x^2 + y^2 = 5^2 = 25$$

Notice that x and y *change* as the ladder slides downward, while the length of the ladder remains constant regardless of time. If we differentiate with respect to t, we get

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(25)$$
$$\implies 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$



This is a *differential equation* which describes the motion of the ladder at any instant. $\frac{dx}{dt}$ is the rate at which the bottom of the ladder is sliding away from the wall, and $\frac{dy}{dt}$ is the rate at which the top of the ladder slides toward the ground.

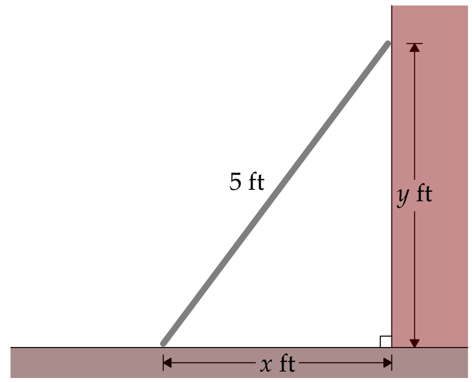
Problems involving differential equations with respect to time t are called *related rates* problems. We will be asked to find the rate of change of some quantity with respect to time. Here is the method:

1. Draw a picture to model the situation described in the problem.
2. Write down important information provided by the question and label the picture. Be sure to distinguish between *variables* and *constants*!
3. Write an equation or multiple equations connecting the introduced variables. Often you will need to use area formulas, volume formulas, etc. to write the equations.
4. Differentiate with respect to t (implicit!) to obtain a differential equation.
5. Make suitable substitutions to solve for the case described in the question.

As a warning, be sure not to substitute in values for the described case too early! Otherwise we will wrongly treat variables as constants. We need the differential equation in its most general form first.

Also be cognizant of what each of your defined variables represents. This is the key for understanding its derivative and its units.

Examples 3.36. 1. We return to Example 3.35. Suppose a 5 ft ladder slides such that the bottom of the ladder moves away from the wall at a constant rate of 2 ft/s. How fast is the top of the ladder moving toward the ground in the instant when the top is 3 ft from the ground?



We use the picture previously drawn. In this case, x and y are variables; x represents the distance between the bottom of the ladder and the wall, and y represents the distance between the top of the ladder and the wall. The length of the ladder is constant, so we can write

$$x^2 + y^2 = 25$$

Now we can implicitly differentiate with respect to t:

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

Notice what the question is asking: we want to know how fast the top of the ladder is moving toward the ground at a particular instant. That is, we want the *rate of change* of the distance between the top of the ladder and the ground. What we want is $\frac{dy}{dt}$!

In this instant, $y = 3 \Longrightarrow x = 4$ from the Pythagorean Theorem. Also, since the bottom of the ladder moves away from the wall at 2 ft/s, that means the distance between the bottom and the wall is increasing by 2 ft/s. Since x is the distance between the bottom of the ladder and the wall, then its *derivative* with respect to t must be the rate of change of that distance. So $\frac{dx}{dt} = 2$ at this time!

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0 \Longrightarrow 2(4) \cdot (2) + 2(3) \cdot \frac{dy}{dt} = 0$$
$$\Longrightarrow 16 + 6 \cdot \frac{dy}{dt} = 0 \Longrightarrow \frac{dy}{dt} = -\frac{16}{6} = -\frac{8}{3}$$

As a sanity check, notice that the rate of change of y is negative. We should certainly expect this, since the distance between the top of the ladder and the ground is decreasing as time moves!

Since the unit of y is feet and the unit of t is seconds, the rate of change of the distance between the top of the ladder and the ground is $-\frac{8}{3}$ ft/s. We would answer by saying the top of the ladder is moving toward the ground at $\frac{8}{3}$ ft/s (the negative is redundant).

As a comment, note that we realized the hypotenuse (length of the ladder) is constant and labeled it '5 ft' in our picture and in our equation. Some students may mistakenly also label the hypotenuse as a variable, say z. This is not a crucial mistake here, since we would have gotten

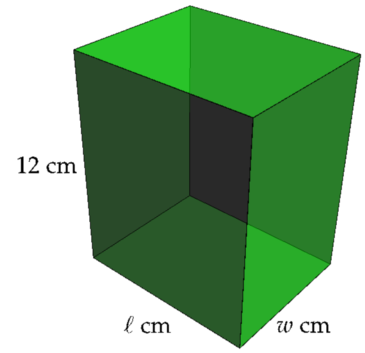
$$x^2 + y^2 = z^2 \Longrightarrow 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2z \cdot \frac{dz}{dt}$$

and realized that $\frac{dz}{dt} = 0$ since z represents the length of the ladder, and hence the right side of the equation would have vanished anyway. But it is a major mistake to classify a variable as a constant, e.g.

$$4^2 + 3^2 = 5^2$$

would leave us nothing to differentiate!

2. A rectangular box has a height of 12 cm. Its length is increasing at a constant rate of 3 cm/s, and its width is decreasing at a constant rate of 6 cm/s. How fast are the volume and surface area of the box changing when the length of the box is 10 cm and the width is 8 cm?



We draw a box first, labeling w cm for width and ℓ cm for length. Since the question simply specifies that the height of the box is 12 cm, we can label the height as 12 cm, since it is *constant*. What we want are the rates of change of *volume* and *surface area*, so we can write the equations for the volume and surface area of a rectangular prism:

$$V = \ell w h = 12\ell w$$
$$A = 2\ell w + 2\ell h + 2wh = 2\ell w + 24\ell + 24w$$

Now differentiate (we need the product rule here):

$$\frac{dV}{dt} = 12\left(\frac{d\ell}{dt}\cdot w + \ell\cdot\frac{dw}{dt}\right) \qquad \frac{dA}{dt} = 2\left(\frac{d\ell}{dt}\cdot w + \ell\cdot\frac{dw}{dt}\right) + 24\frac{d\ell}{dt} + 24\frac{dw}{dt}$$

Now we substitute the values in the particular case. We plainly see in this moment, $\ell = 10$ cm, $w = 8$ cm, and $\ell' = 3$ cm/s. Since w is *decreasing*, we set $w'(t) = -6$ cm/s. Remember, our goal is to find the rate of change of volume $V'(t)$ and surface area $A'(t)$ in this instant:

$$\frac{dV}{dt} = 12\,(3\cdot 8 + 10\cdot -6) = -432\,\text{cm}^3/\text{s}$$
$$\frac{dA}{dt} = 2\,(3\cdot 8 + 10\cdot -6) + 24(3) + 24(-6) = -144\,\text{cm}^2/\text{s}$$

Note the unit cm^3/s, since the units for volume are cubic lengths. We can now say that the volume of the box is decreasing at a rate of $432\,\text{cm}^3/\text{s}$ in this moment.

And for surface area, since units for area are square lengths, the surface area of the box is decreasing at a rate of $144\,\text{cm}^2/\text{s}$.

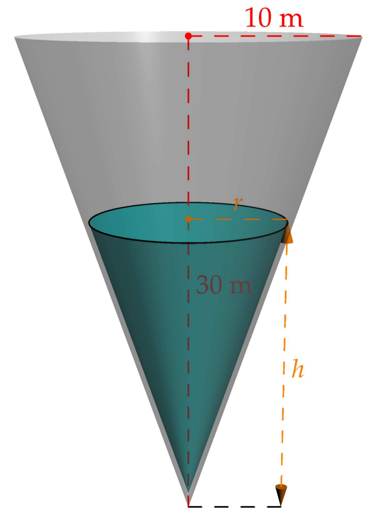
Again, don't be fooled into labeling the edge lengths for w and ℓ prematurely. In the previous example, we made sure to label the hypotenuse of the triangle 5 ft since it was constant, but even if we didn't, we would have gotten the same answer by doing the correct substitutions.

But in this example, say we did not label $h = 12\,cm\ldots$

$$V = \ell w h = \ell[wh]$$
$$\implies \frac{dV}{dt} = \frac{d\ell}{dt}\cdot(wh) + \ell\cdot\left(\frac{dw}{dt}\cdot h + w\cdot\frac{dh}{dt}\right)$$
$$\implies \frac{dV}{dt} = 3(8)(12) + 10(-6(12) + 8(0)) = -432\,\text{cm}^3/\text{s}$$

which indeed gave us the same answer as previous by realizing $h'(t) = 0$, though is this multiple product rule nonsense ever something you want to compute?!

3. An inverted right circular conical water tank is draining from from the bottom such that the depth of water decreases at a constant rate of 2 m/h. The full depth of the cone is 30 m, and the radius of the base of the cone is 10 m. Find the rate at which the volume is changing at the instant when the depth of the water is $h = 18$ m.



We start by drawing and labeling an inverted cone. We note that the full depth of the cone is constant at 30 m, and the radius at the base is 10 m. However, the water itself is also in a conical shape and changes over time, so we label the depth of water h and the radius r.

An important fact about cones is that the radius and height are always in *constant ratio* at any cross section parallel to the base. For this cone, since the full height is 30 and the radius of the base is 10, we have $h = 3r$. We will return to this later; for now, the volume of a cone is:

$$V = \frac{1}{3}\pi r^2 h \Longrightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left(2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \cdot h + r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right)$$

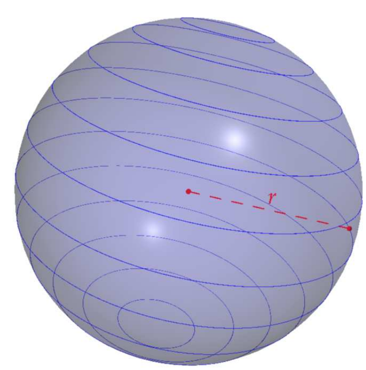
Our goal is to find $V'(t)$ as specified by the question. In the moment that the water's depth is $h = 18$ m, the radius is $r = 6$ m since $h = 3r$. We are given the depth of water decreases at 2 m/h$\Longrightarrow h'(t) = -2$. But how do we get $r'(t)$? Again, we use

$$h = 3r \Longrightarrow \frac{\mathrm{d}h}{\mathrm{d}t} = 3\frac{\mathrm{d}r}{\mathrm{d}t} \Longrightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{2}{3}$$

whence we needed a *second equation* to find all the required information. Finally,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left(2(6)\left(-\frac{2}{3}\right)(18) + (6)^2(-2)\right) = -72\pi\,\mathrm{m}^3/\mathrm{h}$$

4. Air is pumped into a spherical yoga ball at a constant rate of 4 mm^3 per minute. How fast is the surface area of the balloon increasing at the moment when the radius is 3 mm?



Since the radius, volume, and surface area of the sphere change over time, we note that none are constants when drawing and labeling our figure. The sphere's volume is

$$V = \frac{4}{3}\pi r^3 \Longrightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \Longrightarrow 4 = 4\pi(3)^2 \cdot \frac{\mathrm{d}r}{\mathrm{d}t}$$

So we get that, in this instant, the rate of change of the radius is $r'(t) = \frac{1}{9\pi}$ mm/min.

Now we use the surface area formula:

$$A = 4\pi r^2 \Longrightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} = 8\pi(3)\left(\frac{1}{9\pi}\right) = \frac{8}{3}\,\mathrm{mm}^2/\mathrm{min}$$

Again, note that in this example, we used two equations to get all necessary information for our desired rate of change. Related rates problems require some ingenuity and practice to nail!

Exercises 3.5. 1. A 13 foot ladder rests against a vertical wall with its feet on the ground. The the ladder slips, and the instant when the top of the ladder is 5 feet from the ground, it is moving at 8 feet per second. How fast is the bottom of the ladder moving away from the wall in this moment?

2. The variables a and b are related by the equation $ab^2 = 100$. At the instant when $a = 6$, b is increasing by 2 units per second. What is the rate of change of a at this moment?

3. Water is spilled on the floor and spreads in a circular fashion. The radius of this spill grows at a constant rate of 3 centimeters per second. How fast is the area of the spill increasing when its radius is 50 centimeters?

4. Gravel is being poured from a truck onto the ground, and the gravel collects in a conical shape such that the diameter of the base of the pile is always eight times the height of the pile. If the gravel is being deposited at a constant rate of 50 cubic inches per second, how fast is the height of the gravel pile increasing when the diameter is 24 inches?

5. Adam begins running north at a constant speed of 8 miles per hour. At a later time, Bob begins running east from the same starting position at a constant speed of 6 miles per hour. At what rate is the distance between Adam and Bob changing when Bob is 7 miles from the starting position and 16 miles apart from Adam?

6. The length of a rectangle is decreasing at 2 mm per minute. However, the area of the rectangle always remains constant at $400\,\text{mm}^2$. At what rate is the width changing at the instant when the rectangle is a square?

7. A particle travels in the xy-plane along the curve $e^y = 2x + x^2$ such that its x-coordinate is changing at a constant rate of 3 units per second. Find the rate of change of the y-coordinate of the particle when $x = 5$.

8. (Calculator) Two airplanes A and B leave the same airport simultaneously at 120° to each other, with constant speeds of 880 km/h and 910 km/h respectively. Find the rate at which the distance between them is changing after two hours.

   (*Hint: Recall the law of cosines* $c^2 = a^2 + b^2 - 2ab\cos C$)

9. A square is inscribed within a circle whose radius is decreasing at a constant rate of 8 units per second. At what rate is the area of the inscribed square decreasing when the side lengths of the square are 5 units?

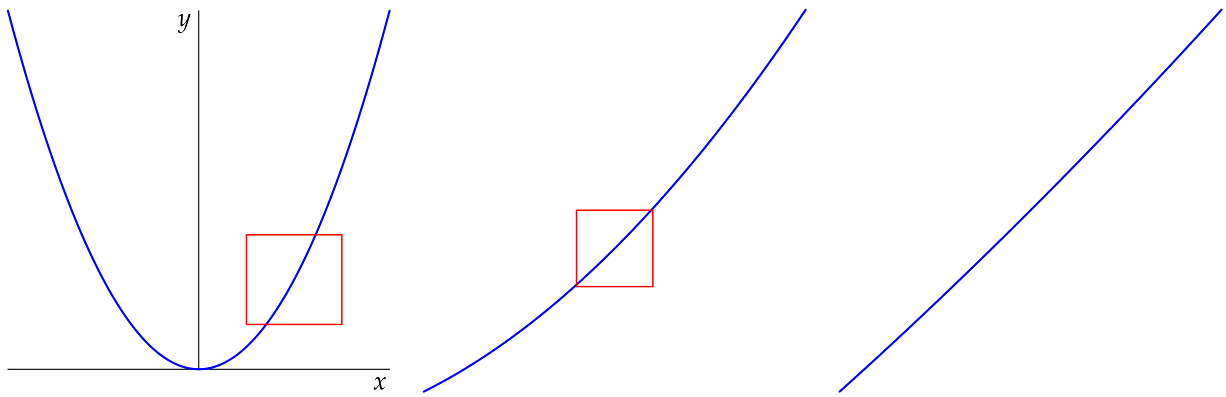
10. The ideal gas law for a particular quantity of gas says that $PV = T$, where P is pressure in pascals (Pa), V is volume in cubic meters, and T is temperature in Kelvin (K). At a given time, suppose that the gas is being heated at a rate of 3 K/s, and the volume reduced at $0.01\,\text{m}^3/\text{s}$. If, at this time, the volume is $2\,\text{m}^3$ and the pressure is 10000 Pa, find the rate of change of the pressure.

11. (Hard) A right triangle's leg has a constant length of 6 units. The length of the hypotenuse is decreasing at a constant rate of 1.5 units per second. In the instant when the length of the hypotenuse is 12 units, what is the rate of change of the *angle* opposite the constant side?

    (*Hint: The units for your answer should be radians per second.*)

3.6 Linear Approximation and Differentials

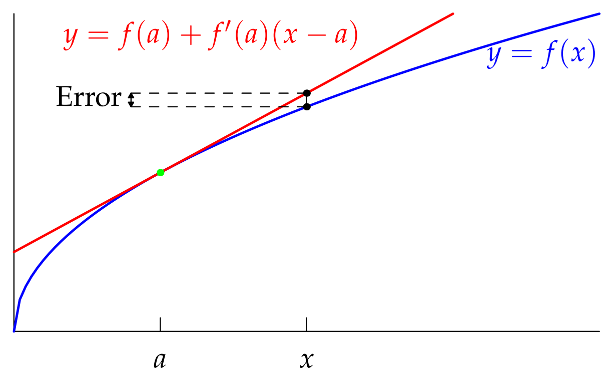
We begin this section with an exploration. Take a graphing calculator and graph the function $y = x^2$, then use the 'zoom' feature several times at any point. We should get something similar to this:



In fact, this does not have to be done with the graph of $y = x^2$. For most 'nice' graphs that we choose, as we zoom in continuously at any point, the graph begins to look *linear*. This is indeed an important property: locally, *smooth* functions behave *linearly*. We use this to approximate values of differentiable functions.

Here is the general idea: we use the tangent line to the original curve that we are familiar with by now at some point $x = a$ to estimate nearby values of that function.

The *error* is the difference between the approximate and correct y-values. Importantly, we should select x-values *nearby* in order to yield more accurate estimations, i.e. decrease error.



Definition 3.37. The *linear approximation* or *local linearization* of $y = f(x)$ at the point $(a, f(a))$ is the tangent line to f through that point:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

Sometimes we may write $L_a(x)$ to highlight that the approximation is near $x = a$.

Example 3.38. Use a tangent line approximation of $g(x) = \sqrt{x}$ at $x = 4$ to approximate $\sqrt{4.2}$.
We are being asked to approximate $g(4.2) = \sqrt{4.2}$. We find the tangent line equation at $x = 4$ first:

$$g(4) = \sqrt{4} = 2 \quad \text{and} \quad g'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \Longrightarrow g'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

where the linear approximation is

$$L(x) = g(4) + g'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) \Longrightarrow \sqrt{4.2} \approx L(4.2) = 2 + \frac{1}{4}(4.2 - 4) = 2.05$$

Localism

The linear approximation is only useful *locally*. Recall our discussion earlier: the graph of an arbitrary differentiable function only appears like a line if we zoom in very closely. As a result, the approximation $f(x) \approx L_a(x)$ is good when x is close to a, and generally gets *worse* as the distance between x and a gets larger. When the difference between x and a is large, $L_a(x)$ becomes useless, so best to use a tangent line to the curve at another point.

Examples 3.39. 1. Without a calculator, use a local linearization of $f(x) = \sin x$ to approximate the value of $\sin 3$. You may assume $\pi = 3.14$ for your calculations.

We choose a value near $x = 3$ to produce a good approximation. By noticing that $\pi \approx 3.14$, we use the tangent line to f at $x = \pi$:

$$f(\pi) = \sin\pi = 0 \quad \text{and} \quad f'(x) = \cos x \Longrightarrow f'(\pi) = \cos\pi = -1$$

Therefore, our linear approximation is

$$f(x) \approx L_\pi(x) = f(\pi) + f'(\pi)(x-\pi) = -(x-\pi) \Longrightarrow \sin 3 \approx L_\pi(3) = -(3-\pi) = 0.14$$

We could have instead used the linear approximation centered at $x = \frac{\pi}{2}$, which is another nice value of $\sin x$. In this case we obtain

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1 \quad \text{and} \quad f'(x) = \cos x \Longrightarrow f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

and the linear approximation in this case would be

$$L_{\frac{\pi}{2}}(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) = 1 \Longrightarrow \sin 3 \approx L_{\frac{\pi}{2}}(3) = 1$$

Clearly, the latter is a terrible approximation! If you ask your calculator, $\sin 3 = 0.1411$ to four decimals, which is remarkably close to our first estimation. The crucial observation is that 3 is much closer to π than to $\frac{\pi}{2}$, and as a result, the linear approximation using $a = \pi$ was much more accurate.

2. Example 3.38 cont. Recall that we used $a = 4$ to do our approximation. To contrast, suppose instead that we chose to use the tangent line to $g(x) = \sqrt{x}$ at $x = 9$, another perfect square:

$$g(9) = \sqrt{9} = 3 \quad \text{and} \quad g'(x) = \frac{1}{2\sqrt{x}} \Longrightarrow g'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Thus our approximation for this choice of a is

$$g(x) \approx L_9(x) = 3 + \frac{1}{6}(x-9) \Longrightarrow \sqrt{4.2} \approx L_9(4.2) = 3 + \frac{1}{6}(4.2-9) = 2.2$$

Since 4.2 is much closer to 4 than to 9, we would expect that the approximation $L_4(4.2) = 2.05$ is more accurate than the approximation $L_9(4.2) = 2.2$. Indeed, if you use your calculator, we get $\sqrt{4.2} = 2.0494$ to four decimals, demonstrating that the first estimate is superior.

Error

Definition 3.40. The *error* of a linear approximation centered at a for $x = c$ is the difference between the estimate and the true value

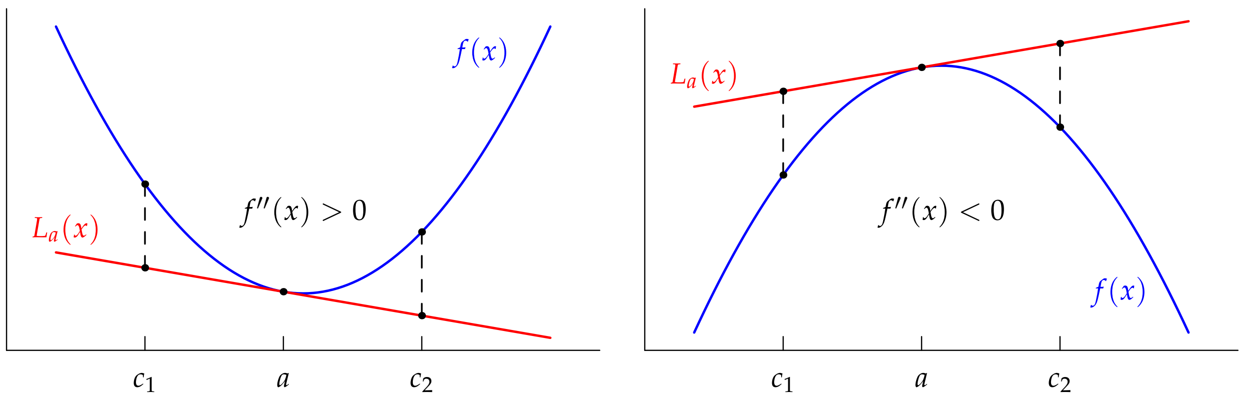
$$\text{Error} = L_a(c) - f(c)$$

We sometimes say *absolute error* to indicate we want the absolute value of the difference $|L_a(c) - f(c)|$.

If $L_a(c) < f(c)$, then $L_a(c)$ is an *underestimate* or *under-approximation* for $f(c)$.

If $L_a(c) > f(c)$, then $L_a(c)$ is an *overestimate* or *over-approximation* for $f(c)$.

Obviously, an error closer to zero indicates a better approximation.
The definitions for under- and over-approximation should be clear, but how would we determine if our estimate is greater than or less than the real value? We observe the following:



The graph of f is *concave up*
$L_a(x) < f(x)$

The graph of f is *concave down*
$L_a(x) > f(x)$

Regardless of whether f is increasing or decreasing, the *concavity* of the graph of f is what determines whether a linear approximation underestimates or overestimates a function value.

Theorem 3.41. Let f be a twice-differentiable function and $L_a(x)$ the tangent line to f at $x = a$.

If $f''(x) > 0 \iff f$ is concave up on the open interval between a and c, then $L_a(c) < f(c)$. Otherwise said, $L_a(c)$ underestimates $f(c)$.

If $f''(x) < 0 \iff f$ is concave down on the open interval between a and c, then $L_a(c) > f(c)$. Otherwise said, $L_a(c)$ overestimates $f(c)$.

Examples 3.39 cont. 1. Recall $f(x) = \sin x$. We have

$$f'(x) = \cos x \implies f''(x) = -\sin x$$

We could construct a second derivative sign chart to see that $f''(x) > 0 \iff f$ is concave up on the open interval $3 < x < \pi$. From Theorem 3.41, we should expect that $L(3) < f(3)$, or that the tangent line *under-approximates* the true value of $f(3)$. Indeed, we had $L_\pi(3) = 0.14$ and $\sin 3 = 0.1411$ (to four decimals).

2. For $g(x) = \sqrt{x}$, we have

$$g'(x) = \frac{1}{2}x^{-1/2} \implies g''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt{x^3}} < 0 \text{ for all } x > 0$$

$g''(x) < 0 \iff g$ is concave down for all x on its domain, so *any* tangent line approximation produces an overestimate! Just to check, we had $L_4(4.2) = 2.05$ and $g(4.2) = \sqrt{4.2} = 2.0494$ (to four decimals), which is indeed an overestimate.

Do note that if the graph of f changes concavity i.e. has a point of inflection on the interval between a and c, then Theorem 3.41 is unreliable, but it is rare for us to need to consider such cases.

Differentials

By now we are used to interchangeably writing $f'(x) = \frac{dy}{dx}$. Remember why Leibniz's notation for derivatives is presented as a fraction representing change in y with respect to x:

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

where we are viewing $\Delta x = x - a$ as a tiny change in x which *causes*, via the function f, a corresponding change $\Delta y = f(x) - f(a)$ in the value of f. If we view dy and dx as *extremely small changes* in x and y respectively, then we may write

$$f'(x) = \frac{dy}{dx} \implies dy = f'(x)dx$$

What does this mean? If $x = a$ and we increase x by an infinitesimally small amount dx, then y will increase by an infinitesimally small amount $dy = f'(a)dx$.

Definition 3.42. The expressions dx and dy[18] are called *differentials*, where dy represents the *approximate change* in $y = f(x)$ in response to a small change in x, which is dx:

$$dy = f'(x)dx$$

Differentials become helpful when we are interested only in how much a function value changes when its input changes by a small amount.

Examples 3.43. 1. If $y = x^2 - 2x$, find dy and evaluate when $x = 3$ and $dx = 0.1$.

If $y = f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$ and thus

$$dy = (2x-2)dx$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2(3)-2)(0.1) = 0.4$$

[18] Previously, we had used the expressions dy and dx in Leibniz's derivative notation $\frac{dy}{dx}$, but they did not have any meaning on their own. This is one of the few instances in this course where we will see dy and dx as independent expressions.

2. For a farmer, the annual cost, in dollars, of feeding x cattle per month is given by

$$h(x) = 0.004x^3 - 0.06x^2 + 100$$

Suppose that the farmer is currently raising 25 cattle this month. If, in the next month the farmer decides to feed 28 cattle, what is the approximate change in cost of feed?

What we need is the change $\mathrm{d}h$ when $x = 25$ changes by $\mathrm{d}x = 3$. First,

$$h'(x) = 0.012x^2 - 0.12x = 0.012x(x-10) \Longrightarrow h'(25) = 0.012(25)(25-10) = 4.50 \text{ dollars}$$

Then the approximate increase in cost when $\mathrm{d}x = 3$ is

$$\mathrm{d}h = h'(25)\mathrm{d}x = 4.50 \cdot 3 = 13.50 \text{ dollars}$$

As with local linearization, the larger the change $\mathrm{d}x$, the worse our estimated change gets. Here, the exact increase in cost, if we use a calculator, is

$$h(28) - h(25) = 15.79 \text{ dollars}$$

Why are we discussing differentials here? Remember that differentials are used to approximate the change in the value of a function caused by a small change in input. Let f be a differentiable function at $x = a$, and suppose that x changes by a small amount. In the context of this conversation, we are concerned with how much y changes (Δy) in response to this change in x. Say $x = a$ changes by an amount $\mathrm{d}x$. Then

$$\Delta y = f(a + \mathrm{d}x) - f(a)$$

However, instead of calculating the exact change in y, we can use *linear approximation*. Remember that if x is close to a, then we have the estimate

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

And if $\mathrm{d}x$ is small, then

$$f(a + \mathrm{d}x) \approx L(a + \mathrm{d}x) = f(a) + f'(a)(a + \mathrm{d}x - a) \Longrightarrow f(a + \mathrm{d}x) - f(a) \approx f'(a)\mathrm{d}x$$

To really spell it out, we have

$$\Delta y = f(a + \mathrm{d}x) - f(a) \approx f'(a)\mathrm{d}x = \mathrm{d}y$$

In essence, finding the estimated change in a function f in response to a small change in $x = a$ is equivalent to performing a linear approximation!

The upshot is that the primary usefulness of differentials comes when we simply want to estimate *error* in a quantity. Suppose that $y = f(x)$, where the value of x is known within some error range $x \pm \mathrm{d}x$. The corresponding potential error in y can be evaluated using the differential $\mathrm{d}y = f'(x)\mathrm{d}x$.

Be warned, the error we are talking about here is not the same as the error we discussed earlier in the section! There, we were referring to the difference between an approximation and the true value of a function. Here, we mean the change in a function value induced by a small potential measurement inaccuracy in input.

Examples 3.44. 1. Suppose the side length of a cube is measured using a ruler and observed to be $s = 10\,\text{cm}$. Thus, the volume of the cube is $V = 10^3 = 1000\,\text{cm}^3$. However, there are only tick marks on the ruler every millimeter, meaning the potential error in the measurement of each side is $\frac{1}{2}$ mm, or $\frac{1}{20}$ cm. What is the resulting potential error in volume?

Since $V = s^3$ and the error $\mathrm{d}s = 0.05\,\text{cm}$, we have

$$\mathrm{d}V = 3s^2\mathrm{d}s = 3 \cdot 10^2 \cdot 0.05 = 15\,\text{cm}^3$$

So the approximate possible error in the volume of the cube is $15\,\text{cm}^3$. So we might say that the volume of the cube is $V = 1000 \pm 15\,\text{cm}^3$.

2. (Calculator) The Body Mass Index (BMI) of a human is $B = m \cdot h^{-2}$, where m is the mass of the subject in kilograms and h their height in meters. Suppose a person's mass is known to be exactly 65 kg, but their height measure fluctuates 1.2 cm=0.012 m above or below 1.78 m. Find the resulting error in the BMI of our subject.

Since $m = 65\,\text{kg}$ exactly, we can write $B = 65h^{-2}$. The possible error in the person's height is 0.012 m greater than or less than 1.78 m, so $h = 1.78$ and $\mathrm{d}h = 0.012$. Thus

$$\mathrm{d}B = -130h^{-3}\mathrm{d}h = -\frac{130}{1.78^3} \cdot 0.012 = -0.277\,\text{kg/m}^2$$

Percentage Error

Students likely have encountered percent error in a science class prior.

Definition 3.45. The *percentage error* of a quantity Q is approximated by its differential divided by its actual value expressed as a percentage:

$$\text{Percentage Error} = \frac{\mathrm{d}Q}{Q} \cdot 100\%$$

Some questions will give the error $\mathrm{d}x$ as a percent or ask for percentage error of $\mathrm{d}y$, so do the necessary conversions to answer the question correctly.

Examples 3.46. 1. In an equilateral triangle, the length of each of its sides is 6 m up to a possible error of 2.5%. Find the approximate possible error in the area of the triangle.

First note that the area of an equilateral triangle with side length x is $A = \frac{\sqrt{3}}{4}x^2$. The percentage error in x if $x = 6$ is

$$2.5\% = 0.025 = \frac{\mathrm{d}x}{6} \implies \mathrm{d}x = 0.025 \cdot 6 = 0.15$$

Thus the error in area A is

$$\mathrm{d}A = \frac{\sqrt{3}}{2}x\mathrm{d}x = \frac{\sqrt{3}}{2} \cdot 6 \cdot 0.15 = \frac{9\sqrt{3}}{20}$$

And, if we were asked, the percentage error in A would be

$$\frac{dA}{A} \cdot 100\% = \frac{9\sqrt{3}}{20} \cdot \left(\frac{\sqrt{3}}{4} \cdot 36\right)^{-1} \cdot 100\% = 5\%$$

2. Example 3.44.2 cont. Suppose that a subject's mass is known to be exactly 44.5 kg, but their height is known to be 1.55 m with a margin of error of 2%. What is the approximate maximum percentage error of this person's BMI?

   This person's weight is exactly 44.5 kg, so $m = 44.5 \implies B = 44.5h^{-2}$. The maximum error of this subject's height is 2% of 1.55 m, so $dh = 1.55 \cdot (0.02) = 0.031$ m. Therefore

$$dB = -89h^{-3}dh = -\frac{89}{1.55^3} \cdot 0.031 = -0.741\,\text{kg/m}^2$$

   and to find the percentage error, we need B:

$$B = m \cdot h^{-2} = 44.5 \cdot 1.55^{-2} = 18.522$$

   And thus the percentage error is

$$\frac{dB}{B} = \cdot 100\% = \frac{-0.741}{18.522} \cdot 100\% = -4.000\%$$

Exercises 3.6. 1. Use the local linearization of $f(x) = \sqrt[4]{x}$ at $x = 16$ to approximate $\sqrt[4]{15}$. Is this value an underestimate or overestimate? Justify your answer.

2. Without a calculator, use a linear approximation of $g(x) = \cos x$ to approximate the value of $\cos 1.5$. You may assume $\pi = 3.14$ for your calculations.

3. Consider the function f defined by

$$f(x) = \frac{1}{3 - x^2}, \quad x \neq \pm\sqrt{3}$$

   (a) Find the equation of the tangent line to f at $x = 1$.

   (b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Is this approximation greater than or less than $f(1.1)$? Explain your reasoning.

4. Consider the natural exponential function $h(x) = e^x$.

   (a) Find the equation of the tangent line to h at $x = 0$.

   (b) Use the tangent line equation from part (a) to approximate $e^{0.2}$.

   (c) Show that *every* linear approximation for $h(x) = e^x$ produces an underestimate.

5. Find the value $x = b$ for which

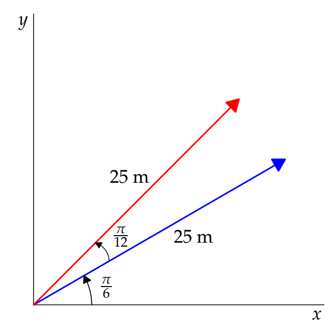
$$\begin{cases} L_{16}(x) & \text{is a better approximation to } \sqrt{x} \text{ for } x < b \\ L_{25}(x) & \text{is a better approximation to } \sqrt{x} \text{ for } x > b \end{cases}$$

6. Consider the function $f(x) = x^3 - 3x$.

   (a) Find the equation of the tangent line at $x = 4$ for f.

   (b) (Calculator) Use this tangent line to approximate $f(5)$, $f(4.1)$, and $f(4.01)$ then compare to the actual values using a calculator.
   (*This should convince you of the importance that the difference between a and x be small!*)

7. (Hard) A projectile is propelled at an angle of $\theta = \frac{\pi}{6}$ with the positive x-axis at a distance of *exactly* 25 m. However, the targeting algorithm is liable to a potential error of up to $\frac{\pi}{12}$.



   (a) Find the potential error in the x-coordinate of the particle's landing position.

   (b) Find the potential error in the y-coordinate of the particle's landing position.

   (c) (Calculator) Find the distance between the particle's intended landing position and its landing position when the angle's error is $\frac{\pi}{12}$, i.e. when $\theta = \frac{\pi}{4}$.

3.7 L'Hôpital's Rule and Indeterminate Forms

We round off this chapter by returning to limits. Recall that most limits were able to be evaluated using substitution and the basic limit laws. Sometimes, however, we run into issues and must use some sneakier methods. For example:

1. We may have to factor components of rational functions. If we try to evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$ by simply substituting in $x = 3$, we get the meaningless result $\frac{0}{0}$, so

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6$$

2. For the Fundamental Trigonometric Limit

$$\lim_{x\to 0} \frac{\sin x}{x}$$

we must use a geometric argument and the Squeeze Theorem.

These limits are examples of *indeterminate forms*: expressions for which attempting to compute the limit by substitution results in a meaningless mathematical expression such as $\frac{0}{0}$. There are other similar expressions.

Definition 3.47. An *indeterminate form* is a limit $\lim_{x\to a} F(x)$, where evaluating $F(a)$ directly gives one of the meaningless expressions

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad \infty-\infty,\quad 0^0,\quad 0^\infty,\quad 1^\infty,\quad \text{etc.}$$

For example,

$$\lim_{x\to 0} \frac{x-\sin x}{x+\tan x}$$

produces an indeterminate form if we try to directly substitute in $x = 0$, and none of our previous methods are of any help. How should we deal with this limit?

Theorem 3.48 (L'Hôpital's Rule[19]). Let f and g be differentiable functions such that $g'(x) = 0$ on an open interval containing $x = a$, except perhaps at $x = a$. Suppose $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the latter limit exists, or is $\pm\infty$. The rule also applies to one-sided limits and limits at infinity.

[19]Also sometimes spelled *L'Hospital's Rule*, pronounced 'loh-pee-TAHL'. The s is silent.

The summary is that if we are attempting to evaluate a limit by substitution produces an indeterminate form, we can *differentiate* the numerator and denominator. Then the original limit is equivalent to the new limit.

Examples 3.49. 1. We will try to evaluate the previous limit we encountered:

$$\lim_{x\to 0}\frac{x-\sin x}{x+\tan x}=\lim_{x\to 0}\frac{1-\cos x}{1+\sec^2 x}=\frac{1-1}{1+1}=0 \qquad (\text{type } \tfrac{0}{0})$$

2. Limits at infinity can also be evaluated using L'Hôpital's Rule, as said:

$$\lim_{x\to\infty}\frac{x+1}{2x-3}=\lim_{x\to\infty}\frac{1}{2}=\frac{1}{2} \qquad (\text{type } \tfrac{\infty}{\infty})$$

3. We may sometimes need *multiple* applications of L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{e^{x^2}}{x^3}=\lim_{x\to\infty}\frac{2xe^{x^2}}{3x^2}=\lim_{x\to\infty}\frac{2e^{x^2}+4x^2e^{x^2}}{6x}=\lim_{x\to\infty}\frac{4xe^{x^2}+8xe^{x^2}+8x^3e^{x^2}}{6}=\infty \qquad (\text{type } \tfrac{\infty}{\infty})$$

Common Mistakes

The rule seems very easy to use, and it is! However, that also means it is very easy to wrongly abuse. Here are some common misapplications of L'Hôpital's Rule.

Examples 3.50. 1. Do not use the quotient rule! Differentiate f and g separately. For example

$$\lim_{x\to\infty}\frac{x^2+4}{x^2-2x}\neq\lim_{x\to\infty}\frac{2x\left(x^2-2x\right)-\left(x^2+4\right)(2x-2)}{\left(x^2-2x\right)^2}$$

2. *Simplify* before applying the Rule. For instance

$$\lim_{x\to\infty}\frac{e^x}{e^{2x}+e^{3x}}=\lim_{x\to\infty}\frac{e^x}{2e^{2x}+3e^{3x}}=\lim_{x\to\infty}\frac{e^x}{4e^{2x}+9e^{3x}}=\cdots$$

is an indeterminate form of type $\frac{\infty}{\infty}$. Using L'Hôpital's Rule without any thought here results in a never-ending chain of limits as above. Instead just factorize: the rule is not necessary!

$$\lim_{x\to\infty}\frac{e^x}{e^{2x}+e^{3x}}=\lim_{x\to\infty}\frac{e^x}{e^x(e^x+e^{2x})}=\lim_{x\to\infty}\frac{1}{e^x+e^{2x}}=0$$

3. The rule only applies to indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. E.g.

$$\lim_{x\to 3}\frac{x}{3-x}\neq\lim_{x\to 3}\frac{1}{-1}=-1$$

The left hand side is *not* an indeterminate form, so the rule does not apply. If we took a moment to view the graph of the function, we would see that the left- and right- side limits are

$$\lim_{x\to 3^-}\frac{x}{3-x}=\infty\neq\lim_{x\to 3^+}\frac{x}{3-x}=-\infty\Longrightarrow\lim_{x\to 3}\frac{x}{3-x}=\text{DNE}$$

In this case the original limit does not exist, so we need not use L'Hôpital's Rule.

4. It is possible for $\lim_{x\to a}\frac{f(x)}{g(x)}$ to exist, but for $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ not to exist. In such a case, the rule is not applicable. For instance

$$\lim_{x\to\infty}\frac{x+\cos x}{x}$$

is an indeterminate form of type $\frac{\infty}{\infty}$. If we attempt to use the rule, we get

$$\lim_{x\to\infty}\frac{x+\cos x}{x}=\lim_{x\to\infty}\frac{1-\sin x}{1}=\text{DNE}$$

This is an incorrect use of L'Hôpital's Rule. Read over Theorem 3.48 again: it only applies if the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists. In this case the latter limit does not exist, and therefore we cannot use the rule. We can use previously discussed methods to evaluate the limit instead:

$$\lim_{x\to\infty}\frac{x+\cos x}{x}=\lim_{x\to\infty}\left(1+\frac{\cos x}{x}\right)=1+0=1$$

where the cosine limit could be found with the Squeeze Theorem.

5. Finally, perhaps the most confusing one:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$$

This appears like a legitimate use of the rule. However, recall that we used the Fundamental Trigonometric Limit itself to prove that $\frac{d}{dx}\sin x=\cos x$. To use this fact to calculate the limit on which it depends is, in logic, called *circular reasoning*. Of course, you may use the rule to remind yourself of the result, but it is unjustified, however unlikely we are to encounter this precise case in a free-response problem.

Other Indeterminate Forms

The remaining indeterminate forms can also be dealt with using L'Hôpital's Rule following some algebraic manipulation.

Definition 3.51. An *indeterminate product* is a limit of the form $\lim_{x\to a} f(x)\cdot g(x)$ where

$$\lim_{x\to a} f(x)=0 \text{ and } \lim_{x\to a} g(x)=\infty$$

Here is how we can deal with these:

$$\lim_{x\to a} g(x)=\infty \Longrightarrow \lim_{x\to a}\frac{1}{g(x)}=0 \Longrightarrow \lim_{x\to a} f(x)\cdot g(x)=\lim_{x\to a}\frac{f(x)}{1/g(x)}$$

which is now an indeterminate form of type $\frac{0}{0}$. Then we can apply L'Hôpital's Rule as previously. We could also consider the indeterminate form of type $\frac{\infty}{\infty}$:

$$\lim_{x\to a} f(x)=0 \Longrightarrow \lim_{x\to a}\frac{1}{f(x)}=\infty \Longrightarrow \lim_{x\to a} f(x)\cdot g(x)=\lim_{x\to a}\frac{g(x)}{1/f(x)}$$

Examples 3.52. 1. We try to convert into a quotient so we have an indeterminate form type $\frac{0}{0}$:

$$\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1/\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{-1}{-\sec^2 x/\tan^2 x}$$
$$= \lim_{x\to\frac{\pi}{2}} \cos^2 x \tan^2 x = \lim_{x\to\frac{\pi}{2}} \sin^2 x = 1$$

2. For this example, we try to get a form $\frac{\infty}{\infty}$ just for variety:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} (-x) = 0$$

Definition 3.53. An *indeterminate difference* is a limit of the form $\lim_{x\to a}(f(x) - g(x))$ where

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$$

The approach with these is to try to combine $f(x) - g(x)$ into a single fraction over a common denominator, which will generally yield an indeterminate form of a simpler type.

Examples 3.54. 1. In the first example, we need to simplify into a single fraction and then apply L'Hôpital's Rule several times:

$$\lim_{x\to 0} \left(\frac{1}{x} - \csc x\right) = \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x \sin x}$$
$$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0} \frac{-\sin x}{2\cos x - x \sin x} = \frac{0}{2-0} = 0$$

2. For trigonometric functions, combining into single fractions typically means using *identities*:

$$\lim_{x\to 0}(\cot x - \csc x) = \lim_{x\to 0} \left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\cos x - 1}{\sin x}$$
$$= \lim_{x\to 0} \frac{-\sin x}{\cos x} = \lim_{x\to 0}(-\tan x) = 0$$

Definition 3.55. An *indeterminate power* is a limit $\lim_{x\to a} f(x)^{g(x)}$ where directly evaluating $f(a)^{g(a)}$ would yield 0^0, ∞^0, or 1^∞.

These are tackled by using algebraic properties of exponentials and logarithms. Since the natural logarithm ln is a continuous function, we can take them through the limit operator.

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln f(x)^{g(x)}} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}$$

This reduces the problem to that of finding the limit of an indeterminate product.

Examples 3.56. 1. For x^x, we can do the following:

$$\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln x}$$

for which we need to find

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} (-x) = 0$$

as we did before. Thus

$$\lim_{x \to 0^+} x^x = e^0 = 1$$

2. We can rewrite $\sqrt[x]{x} = x^{1/x}$, so

$$\lim_{x \to \infty} \sqrt[x]{x} = \lim_{x \to \infty} x^{1/x} = e^{\lim_{x \to \infty} x^{-1} \ln x}$$

for which we need to find

$$\lim_{x \to \infty} x^{-1} \ln x = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

and hence

$$\lim_{x \to \infty} x^{1/x} = e^0 = 1$$

Exercises 3.7. 1. Describe the flaw in the following argument: If $f(x) = e^x$, then

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h}{1} = e^x$$

(*Hint: Refer back to Example 3.50.5.*)

2. Evaluate the following limits, if they exist:

(a) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ (b) $\lim_{x\to\infty} \frac{\ln x}{\ln(1+x^2)}$ (c) $\lim_{x\to-\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(d) $\lim_{x\to 0} \frac{x+\tan x}{x}$ (e) $\lim_{\theta\to\frac{\pi}{2}} \frac{\theta - \pi/2}{\cos\theta}$ (f) $\lim_{x\to 0^+} \ln x \cdot \sin x$

(*Hint: L'Hôpital's Rule may not be applicable or required for some of these limits.*)

3. Recall the *compound interest formula*:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P represents the *principal* or *initial* balance, r the annual interest rate expressed as a decimal, n the number of times interest is compounded per year, t the number of years elapsed, and A the final balance.

Suppose one deposits \$1 into a bank account that offers 100% annual interest rate. Find the total funds in the account after one year if interest is compounded *continuously*. That is, find[20]

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

4. (Hard) Let f, g, and h be twice-differentiable functions that satisfy $f(-1) = g(-1) = 6$. The line $y = 6 + \frac{4}{5}(x+1)$ is tangent to both the graph of f at $x = -1$ and the graph of g at $x = -1$.

(a) Use the equation of the tangent line to the graph of g at $x = -1$ described above to approximate $g(0)$. Is there enough information to conclude that this approximation is an overestimate or underestimate? Explain.

(b) Let p be the function which satisfies $p(x) = f(x) \cdot g(x)$. Find $p'(-1)$.

(c) The function f satisfies $f(x) = \frac{1+x^3}{(h(x))^2 - 16}$ for $x \neq -1$. It is known that $\lim_{x\to-1} f(x)$ can be evaluated using L'Hôpital's Rule. Use $\lim_{x\to-1} f(x)$ to find $h(-1)$ and $h'(-1)$. Show the work that leads to your answers.

(d) Suppose $h(2) = \sqrt{15}$. Argue that there exists a value c for $-1 < c < 2$ such that $f'(c) = 5$.

[20] This is precisely the question that mathematician Joseph Bernoulli was exploring in his derivation of Euler's Number e. Of course, it is unjustified for us to use the method following Definition 3.55 to find this limit since e itself is derived from this limit, but this is a good exercise nonetheless.

4 Integration

4.1 The Area Under a Curve

In this chapter, we begin discussing the second important branch of calculus, which is *integral calculus*. As with differentiation, before we can dive straight into the bulk of the methods, we should take some time to develop some notation and understand the basic concepts.

Riemann Sums

Finding the area underneath a curve has been important to mathematics for much of history. It was absolutely essential for the progression of integral calculus.

The reason we care about area beneath a curve, for practical applications, is because we often care about *accumulation of change*. Briefly, let us consider motion again.

Example 4.1. A particle moves along a horizontal line with a constant velocity of $v(t) = 30\,\text{m/s}$.

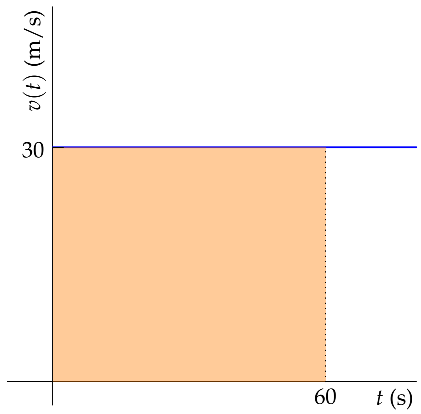
Suppose we want to know how much distance the particle covered over one minute. The most sensible approach is to use our trusty formula

$$\text{rate} = \frac{\text{distance}}{\text{time}} \implies \text{distance} = \text{rate} \cdot \text{time}$$

So, if the particle travels at a constant rate of $30\,\text{m/s}$ for $60\,\text{s}$, then its distance traveled over that time period is

$$30\,\text{m/s} \cdot 60\,\text{s} = 1800\,\text{m}$$

Those observant will notice that 1800 is precisely the *area under the graph* of $v(t)$ from $t = 0\,\text{s}$ to $t = 60\,\text{s}$.



This extends to other rate graphs; the area underneath a derivative graph on an interval represents the *net change* in the value of the original function over that interval.

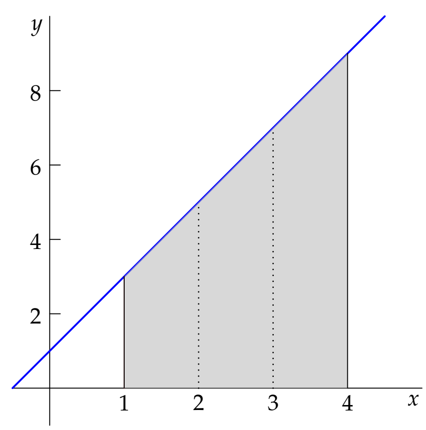
Now that we know the significance of finding area under a curve, we need to establish a method for doing so.

Example 4.2. Say we want to find the area enclosed by the graph of $y = 2x + 1$, the x-axis, and the vertical lines $x = 1$ and $x = 4$.

In this situation, it is quite easy to find this area via geometric methods. The shaded figure is a trapezoid whose parallel bases have lengths 3 and 9, with a height of 4, and thus the area of the bounded region is

$$\text{Area} = \frac{1}{2}(b_1 + b_2) \cdot h = \frac{1}{2}(3 + 9) \cdot 4 = 24 \text{ units}^2$$

The area can be calculated through other geometric means.

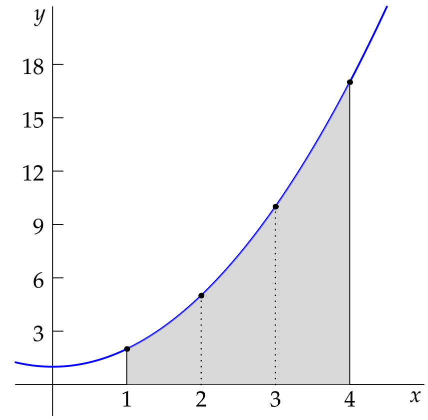


As with rates of change, this seems quite simple. But of course, things get trickier when we consider areas of regions whose boundaries are *curved*.

Consider the function $f(x) = x^2 + 1$, whose graph is shown alongside.

We want to find the area beneath the graph of $y = f(x)$ but above the x-axis, on the interval $[1, 4]$.

In this situation, it is much harder to find the exact bounded area. We can still attempt to rely on geometric methods, but we will not yet be able to get the precise area.



Definition 4.3. Let f be a function defined on a closed interval $[a, b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ a *partition* of $[a, b]$, that is $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

A *Riemann sum* S of f over $[a, b]$ with partition P is

$$S = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and x_i^* is some x-value within the *sub-interval* $[x_{i-1}, x_i]$.

Theorem 4.4. If the interval $[a, b]$ is partitioned into n number of sub-intervals of equal length, then each sub-interval has length

$$\Delta x = \frac{b - a}{n}$$

If you're confused by this mess of mathematical language, a Riemann sum is simply an estimation of the area under a curve between $x = a$ and $x = b$ using the area of n number of *rectangles*, whose height is some value of f within the *sub-interval*. The length of each sub-interval is the length of the *base* of each rectangle.

The reasoning for Theorem 4.4 should be obvious.

Definition 4.5. Depending on our choice of x_i^*, we can produce different Riemann sums:

If $x_i^* = x_{i-1}$ for all i, the method is the *left rule* and gives a *left Riemann sum*.

If $x_i^* = x_i$ for all i, the method is the *right rule* and gives a *right Riemann sum*.

If $x_i^* = (x_{i-1} + x_i)/2$, the method is the *midpoint rule* and gives a *midpoint Riemann sum*.

The definitions of each type of Riemann sum should become clear with examples.

Typically, we are asked most of the time to find one of these three (and one more later) sums. If it helps, construct a table of values! You are not required to sketch the graph of the function or the rectangles, but it may be a good visual guide.

Examples 4.6. For each of the following examples, we are examining the area beneath the graph of $f(x) = x^2 + 1$ but above the x-axis, on the interval $[1, 4]$.

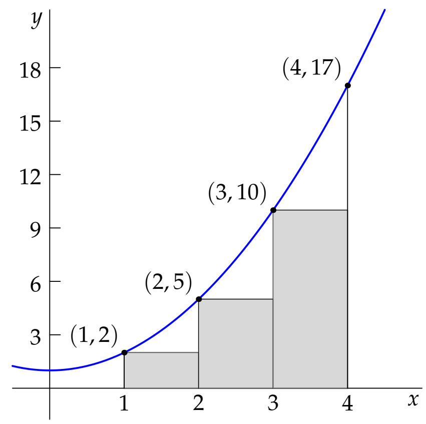
1. We use a *left Riemann sum* to approximate the area of the region using 3 sub-intervals of equal length.

   Each sub-interval should have length

   $$\Delta x = \frac{4-1}{3} = 1$$

   so each rectangle we use to approximate will have a base of 1. A table of values may help for the heights:

x	1	2	3	4
$f(x)$	2	5	10	17

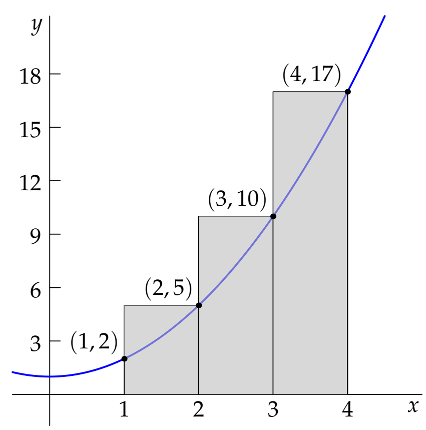
   Since we are using left endpoints, the heights of each rectangle are $f(1) = 2$, $f(2) = 5$, and $f(3) = 10$, and we do not include $f(4)$. So our approximate area is

   $$\begin{aligned} \text{Area} &\approx 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) \\ &= 2 + 5 + 10 = 17 \text{ units}^2 \end{aligned}$$

2. Now use a *right Riemann sum* to estimate the area of the same region using 3 sub-intervals of equal length.

   Each sub-interval similarly has length $\Delta x = 1$, and hence the length of the base of each rectangle is 1. Just to repeat the table of values:

x	1	2	3	4
$f(x)$	2	5	10	17

   This time, since we are using right endpoints, the heights of the rectangles are $f(2) = 5$, $f(3) = 10$, and $f(4) = 17$ and we exclude $f(1)$. Therefore our approximate area is

   $$\begin{aligned} \text{Area} &\approx 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) \\ &= 5 + 10 + 17 = 32 \text{ units}^2 \end{aligned}$$

   And just to note, for Riemann sums where each sub-interval has equal length, we can factor out the length of the base of each rectangle:

   $$\text{Area} \approx 1 \cdot (f(2) + f(3) + f(4))$$

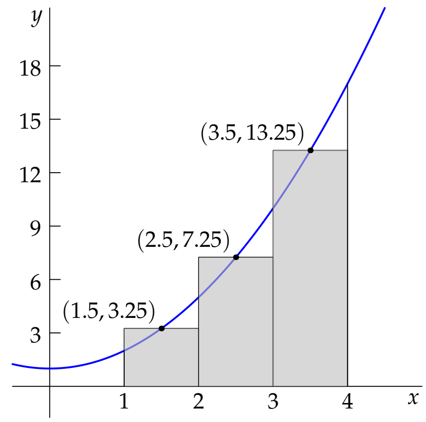
3. Now we use a *midpoint Riemann sum* with the same other details.

   The bases will still each have length $\Delta x = 1$, but the heights now use the *midpoints* of each sub-interval:

x	1.5	2.5	3.5
$f(x)$	3.25	7.25	13.25

   So our approximate area with midpoints is

$$\begin{aligned}\text{Area} &\approx 1 \cdot (f(1.5) + f(2.5) + f(3.5)) \\ &= 3.25 + 7.25 + 13.25 = 23.75 \text{ units}^2\end{aligned}$$



Definition 4.7. When we use the *trapezoidal rule* to find a *trapezoidal sum*, we take the average of the left and right Riemann sums. That is, for each sub-interval $[x_{i-1}, x_i]$, we treat $\Delta x = x_i - x_{i-1}$ as the *height* of a trapezoid, and treat $f(x_{i-1})$ and $f(x_i)$ as the parallel *bases*.

While not explicitly a Riemann sum since it does not use rectangles to approximate, the trapezoidal rule is often quite a good estimate for area.

Examples 4.6 cont. 4. Now we use a *trapezoidal sum* for the aforementioned region.

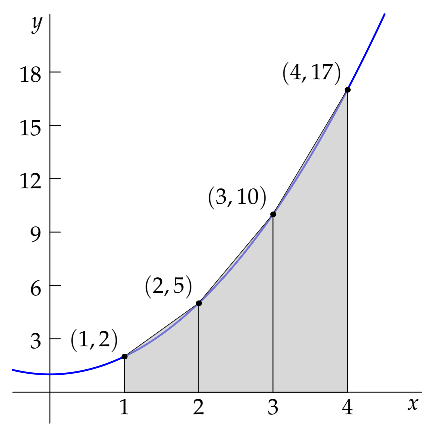
We can use our previously constructed table of values for the left and right sums:

x	1	2	3	4
$f(x)$	2	5	10	17

Our calculations will change slightly; this time, the values of f are acting as the length of the bases of each figure, and the heights are $\Delta x = 1$. Try to shift your perspective if this is difficult to see in the picture.

For the sum, we need the area of each trapezoid:

$$\begin{aligned}\text{Area} \approx &\frac{1}{2}(f(1) + f(2)) \cdot 1 + \frac{1}{2}(f(1) + f(2)) \cdot 1 \\ &+ \frac{1}{2}(f(1) + f(2)) \cdot 1 \\ &= \frac{1}{2}(f(1) + 2f(2) + 2f(3) + f(4)) \\ &= \frac{1}{2}(2 + 2(5) + 2(10) + 17) = 24.5 \text{ units}^2\end{aligned}$$



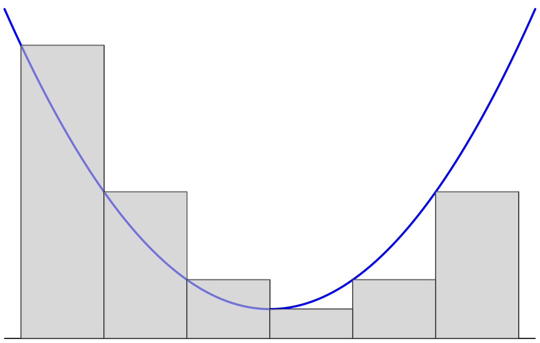
whence 24.5 is precisely the mean of 17 and 32, the left and right Riemann sums respectively.

For our previous examples, if we examine each picture, it seems that the midpoint and trapezoidal sums seemed to be decent approximations of the true area under the graph of $f(x) = x^2 + 1$ on $[1, 4]$ using three equal sub-intervals. The left and right Riemann sums were quite poor estimates, though; the left sum seems to grossly underestimate the true area, and the right sums grossly overestimates. It turns out we can figure out if a right and left Riemann sum underestimates or overestimates area without precisely knowing what the graphs look like.

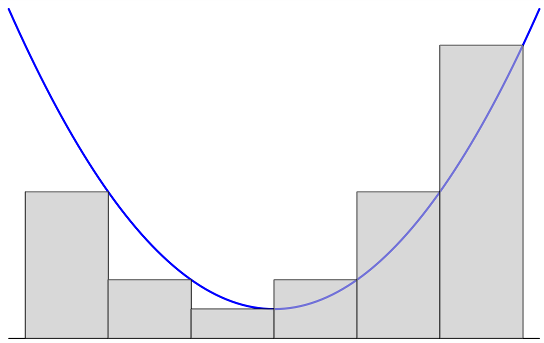
Theorem 4.8. Let f be a differentiable function on $[a, b]$.

If f is increasing on $[a, b]$, then a left Riemann sum underestimates the area under the graph while a right Riemann sum overestimates the area.

If f is decreasing on $[a, b]$, then a right Riemann sum underestimates the area under the graph while a left Riemann sum overestimates the area.



left sum



right sum

Equipped with this Theorem, we can determine if a Riemann sum over- or under-approximates an area with just the equation of $f(x)$ itself, or with information provided in a question.

Examples 4.6 cont. The function $f(x) = x^2 + 1$ has

$$f'(x) = 2x \implies f'(x) > 0 \text{ for } x > 0$$

Since f' is positive on our interval of interest $[1, 4] \implies f$ is increasing, we can conclude that a right Riemann sum would overestimate the area under the curve while a left Riemann sum underestimates that area, which is is consistent with our earlier exploration.

It is not always the case that the interval $[a, b]$ is partitioned evenly for our Riemann sum! This is most often the case when we are provided with a table of values to make an approximation. This just means we must find each Δx individually and we are not allowed to factor our the length of the bases or heights when calculating the sum.

Examples 4.9. Selected values for a differentiable function f are shown in the table:

x	-3	-1	2	3	7	9	14
$f(x)$	4	10	3	7	12	15	9

Use a left, right, and trapezoidal sum to approximate the area under the graph of $y = f(x)$ from $x = -3$ to $x = 14$.

1. Let L represent the left Riemann sum. We have

$$\begin{aligned} L &= \overbrace{(-1+3)}^{\Delta x_1} \cdot \overbrace{4}^{f(-3)} + (2+1)\cdot 10 + (3-2)\cdot 3 + (7-3)\cdot 7 + (9-7)\cdot 12 + (14-9)\cdot 15 \\ &= 168 \text{ units}^2 \end{aligned}$$

2. Let R represent the right Riemann sum. We have

$$\begin{aligned} R &= \overbrace{(-1+3)}^{\Delta x_1} \cdot \overbrace{10}^{f(-1)} + (2+1)\cdot 3 + (3-2)\cdot 7 + (7-3)\cdot 12 + (9-7)\cdot 15 + (14-9)\cdot 9 \\ &= 159 \text{ units}^2 \end{aligned}$$

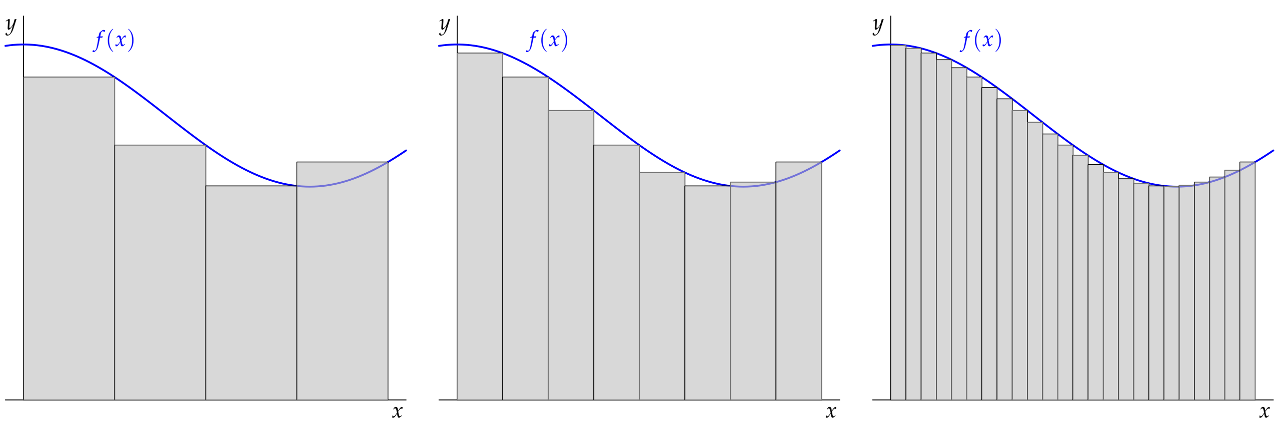
3. Let T represent the trapezoidal sum. Even though we can't factor out each Δx_i, we can at least take out the $\frac{1}{2}$ for each trapezoid's area formula.

$$\begin{aligned} T &= \frac{1}{2}\Big(\overbrace{(-1+3)}^{\Delta x_1} \overbrace{(4+10)}^{f(-3)+f(-1)} + (2+1)(10+3) + (3-2)(3+7) + (7-3)(7+12) \\ &+ (9-7)(12+15) + (14-9)(15+9) \Big) = 163.5 \text{ units}^2 \end{aligned}$$

The Exact Area Under a Curve

Now that we have down the basic technique for Riemann sums, there are several ways we can improve our approximations. We can choose a better type of sum, i.e. trapezoidal may be better than left or right, etc., or we can *increase the number of sub-intervals*.

One should think that by having a larger number n of rectangles, the estimate gets better and better; indeed this is the case!



$n = 4$ $\quad$ $n = 8$ $\quad$ $n = 24$

This begs the question: what if we divided the interval into *infinitely many* sub-intervals to get infinitely many rectangles? More concisely, find the *limit* of the Riemann sum as $n \to \infty$; this sum then would converge on the *exact area* under the curve!

It is worth pointing out that, while this approach seems simple, there are some questions that may arise from it, for instance:

1. Does this process work for *all* functions?
2. Does it matter how we choose the rectangles? More specifically, should we choose a left, right, or other Riemann sum?

These questions are deeper than meets the eye; we will only be able to partially answer these questions with the following Theorem.

Theorem 4.10. Let f be a *continuous* function defined on the closed, bounded interval $[a, b]$. Then the limit of the Riemann sum with n sub-intervals of equal length $\Delta x = (b - a)/2$ is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

converges to a single value *independent* of the choice of sample points x_i^*.

The proof is quite complicated and requires use of the Squeeze Theorem. Thankfully, the argument is beyond the scope of this course. But, in summary, we can use this exact process for *continuous* functions on closed, bounded intervals, which answers our first question. To our second question, we may choose *any* value within each sub-interval $[x_{i-1}, x_i]$ to act as the height.

The exact area under a curve is so important that we give it a special notation.

Definition 4.11. Let f be a continuous function defined on the closed, bounded interval $[a, b]$. The (signed) area between the graph of f and the x-axis is called the *definite integral* of f from a to b:[21]

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

where the function f being *integrated* is called the *integrand*, and a and b are the *lower* and *upper bounds* (or *lower* and *upper limits*) respectively. If the above limit converges to the same value for every choice of sample points, we say f is *integrable*.

The reason we specify signed is because if the graph of f is below the x-axis, then the value of the integral is *negative* by virtue of the values of $f(x)$ being negative.

If needed, be sure to review *sigma notation* for summation. The following identities[22]may be useful for evaluating definite integrals in this manner, if c is constant:

$$\sum_{i=1}^{n} ca_n = c\sum_{i=1}^{n} a_n, \qquad \sum_{i=1}^{n} c = cn, \qquad \sum_{i=1}^{n} i = \frac{1}{2}n\,(n+1)\,, \qquad \sum_{i=1}^{n} i^2 = \frac{1}{6}n\,(n+1)\,(2n+1)$$

The first identity simply says we can factor out any constant coefficients, and the other three allow us to simplify the sums. In practice, we won't usually consider polynomials of higher degree than 2.

[21]Don't worry too much about what the dx term means exactly. You can let it remind you that we are taking sums of *infinitesimally thin* areas.

[22]You do not have to memorize these identities, but you should be able to *use* them when provided.

Because of Theorem 4.10, we can make the simplest choice where each sample point is the right endpoint of each sub-interval: $x_i^* = x_i \implies f(x_i^*) = f(x_i) = f(a + \Delta x i)$.

Also remember that $\Delta x = \frac{b-a}{n}$, where $b - a$ is *constant*; thus we can factor it out and get

$$\text{Area} = (b-a) \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i)}{n} = (b-a) \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(a + \Delta x i)$$

where $(b - a)$ is simply the length of the interval, which for us, may act as the base of the region, and the limit of the sum is the *average height* of the function f. We will return to this idea later.

Example 4.6 cont. Find the exact area under the graph of $f(x) = x^2 + 1$ from $x = 1$ to 4.

We use the limit in Definition 4.11 and the identities provided below it. First, $\Delta x = \frac{4-1}{n} = \frac{3}{n}$, and since we are using right endpoints $x_i^* = x_i$, we have $f(x_i) = f(a + \Delta x i) = f(1 + \frac{3i}{n})$. Now

$$\begin{aligned}
\int_1^4 \left(x^2+1\right) \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1+\frac{3i}{n}\right)^2 + 1\right) \cdot \frac{3}{n} = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + \frac{6i}{n} + 1 + 1\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \left(\frac{9}{n^2} \sum_{i=1}^{n} i^2 + \frac{6}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 2\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \left(\frac{9}{n^2} \cdot \frac{1}{6} n\,(n+1)\,(2n+1) + \frac{6}{n} \cdot \frac{1}{2} n\,(n+1) + 2n\right) \\
&= \lim_{n\to\infty} 3 \left(\frac{9\left(2n^3+3n^2+n\right)}{6n^3} + \frac{6n^2+6n}{2n^2} + \frac{2n}{n}\right) \\
&= 3(3+3+2) = 24 \text{ units}^2
\end{aligned}$$

where the limits in the last part can be evaluated using either the methods discussed in the first chapter or using L'Hôpital's Rule.

Finally, we see that the area of $f(x) = x^2 + 1$ from $x = 1$ to $x = 4$ is *exactly* 24. Compared to our estimations earlier, the midpoint and trapezoidal sum were quite close!

There are two main types of questions involving limits of Riemann sums. The first is evaluating an area using limits, as we did above. The second is *identifying an area written as a limit of Riemann sums*. The key for this second type of question is to look at the equation in Definition 4.11 and equate each of the elements.

Examples 4.12. 1. Rewrite the following limit in integral notation:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2\pi}{n} \cdot \cos\left(\pi + \frac{2\pi k}{n}\right)$$

If we compare to the equation in Definition 4.11, we see that $a = \pi$, $b - a = 2\pi \implies b = 3\pi$, and the function f is $\cos x$. The fact that the index used in the sum is k rather than i does not matter. So, in integral notation, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2\pi}{n} \cdot \cos\left(\pi + \frac{2\pi k}{n}\right) = \int_{\pi}^{3\pi} \cos x \,\mathrm{d}x$$

Alternatively, we could have chosen $a = 0 \implies b = 2\pi$ and let the function f be $\cos(x + \pi)$, which would have yielded

$$\int_0^{2\pi} \cos(x + \pi)\, \mathrm{d}x$$

Often times, it may be easiest to assume $a = 0$ and to look for a suitable expression for $\Delta x = \frac{b-a}{n}$, if that is an option.

2. Identify the definite integral defined by the expression

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\sqrt{\frac{10i}{n}}$$

We can see that $\frac{b-a}{n} = \frac{5}{n} \implies b - a = 5$. Since, inside the square root, there is no additional constant, we may assume $a = 0$ and thus $b = 5$. Note that, since we have $\frac{10i}{n}$ instead of $\frac{5i}{n}$ in the square root, the function f must be $f(x) = \sqrt{2x}$. Thus

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\sqrt{\frac{10i}{n}} = \int_0^5 \sqrt{2x}\, \mathrm{d}x$$

Start getting used to writing the $\mathrm{d}x$ term as well.

In many situations, it is easier to evaluate an integral by using geometry rather than dealing with limits. Look for functions whose graphs produce regions that are triangles, rectangles, circles, etc.

General Properties of Definite Integrals

We can make evaluating definite integrals a bit easier by knowing some of these rules.

Theorem 4.13. Let f and g be integrable on $[a, b]$, and c a constant. Then the following are true:

1. $\displaystyle\int_a^a f(x)\, \mathrm{d}x = 0$
2. $\displaystyle\int_a^b f(x)\, \mathrm{d}x = -\int_b^a f(x)\, \mathrm{d}x$
3. $\displaystyle\int_a^b c\, \mathrm{d}x = c(b - a)$
4. $\displaystyle\int_a^b (f(x) \pm g(x))\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x \pm \int_a^b g(x)\, \mathrm{d}x$
5. $\displaystyle\int_a^b cf(x)\, \mathrm{d}x = c\int_a^b f(x)\, \mathrm{d}x$
6. $\displaystyle\int_a^b f(x)\, \mathrm{d}x = \int_a^c f(x)\, \mathrm{d}x + \int_c^b f(x)\, \mathrm{d}x$

All of these results are easy to confirm by drawing pictures. Think about heights of rectangles, and areas under curves.

The first property says that a line has no area.

Since switching $a \leftrightarrow b$ changes the sign of $\Delta x = \frac{b-a}{n}$, we get the second result.

Since the graph of $f(x) = c$ is a straight horizontal line, the area between the graph of c and the x-axis is a simple rectangle with height c and base $(b-a)$; this is the third property.

The fourth says that the definite integral of a sum or difference of functions is equal to the sum or difference of the definite integrals of those functions independently.

The fifth result stems directly from properties of limits and summations: we can factor out constant coefficients from integral expressions.

The last says we can split an integral through its bounds.

We can use each of these rules to solve problems in which we *don't know* what the functions f, g, etc. are. We only know the value of some of their definite integrals.

Examples 4.14. 1. Suppose we know $\int_{-2}^{5} f(x)\,\mathrm{d}x = 7$. Find $\int_{5}^{-2}(f(x)+1)\,\mathrm{d}x$.

We need to apply the properties above. First, we want to split up the integral:

$$\int_{5}^{-2}(f(x)+1)\,\mathrm{d}x = \int_{5}^{-2} f(x)\,\mathrm{d}x + \int_{5}^{-2} 1\,\mathrm{d}x$$

We also notice that the bounds need to be swapped in order for us to use the given information:

$$\int_{5}^{-2} f(x)\,\mathrm{d}x + \int_{5}^{-2} 1\,\mathrm{d}x = -\int_{-2}^{5} f(x)\,\mathrm{d}x - \int_{-2}^{5} 1\,\mathrm{d}x$$

Now we use the given information:

$$-\int_{-2}^{5} f(x)\,\mathrm{d}x - \int_{-2}^{5} 1\,\mathrm{d}x = -(7) - 1(5-(-2)) = -14$$

2. Suppose $\int_{3}^{9} f(x)\,\mathrm{d}x = -8$, $\int_{9}^{4} f(x)\,\mathrm{d}x = 5$, and $\int_{3}^{4} g(x)\,\mathrm{d}x = 16$. Find $\int_{4}^{3}(2f(x)-g(x))\,\mathrm{d}x$.

First, let's take the definite integral we want the value of and split it up:

$$\int_{4}^{3}(2f(x)-g(x))\,\mathrm{d}x = \int_{4}^{3} 2f(x)\,\mathrm{d}x - \int_{4}^{3} g(x)\,\mathrm{d}x = 2\int_{4}^{3} f(x)\,\mathrm{d}x - \int_{4}^{3} g(x)\,\mathrm{d}x$$

where we used a property to factor 2 out. Now, we don't know what $\int_{4}^{3} f(x)\,\mathrm{d}x$ is, but we can find it from the first two pieces the question gives us:

$$\int_{3}^{9} f(x)\,\mathrm{d}x = \int_{3}^{4} f(x)\,\mathrm{d}x + \int_{4}^{9} f(x)\,\mathrm{d}x \implies -8 = \int_{3}^{4} f(x)\,\mathrm{d}x - 5$$
$$\implies \int_{3}^{4} f(x)\,\mathrm{d}x = -3$$

Now

$$2\int_{4}^{3} f(x)\,\mathrm{d}x - \int_{4}^{3} g(x)\,\mathrm{d}x = -2\int_{3}^{4} f(x)\,\mathrm{d}x + \int_{3}^{4} g(x)\,\mathrm{d}x = -2(-3) + 16 = 22$$

The processes discussed in this section may scare you, but don't worry! Aside from the basic Riemann sums and the properties of definite integrals, most of the limits and such are non-examinable. This analysis is mostly to remind you in subsequent sections that the definite integral is *defined* as an infinite sum of *infinitesimally small* slices. This will aid in our understanding of the applications of integration.

Exercises 4.1. For those who are just reviewing, try not to use the Fundamental Theorem or anti-differentiation for the following problems: that would be to miss the point!

1. Selected values of a differentiable, strictly increasing function g are shown in the table below.

x	-1	1	3	4	5	7	9
$g(x)$	-5	-4	-1	0	1	3	8

   (a) Use the given values in the table to approximate $g'(6)$.

   (b) Using a left Riemann sum, estimate the value of $\int_{-1}^{9} g(x)\,\mathrm{d}x$ with six sub-intervals.

   (c) Is your approximation in part (b) an overestimate or an underestimate of the integral? Explain your reasoning.

   (d) Using a midpoint Riemann sum, estimate the value of $\int_{-1}^{9} g(x)\,\mathrm{d}x$ with three sub-intervals. (*Hint: Use the sub-intervals* $[-1,3]$, $[3,5]$ *and* $[5,9]$. *What are their midpoints?*)

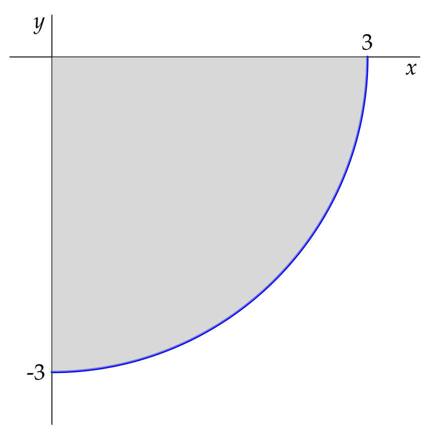
2. Let $y = \sin x$ on $[0,\pi]$. Using a right Riemann sum with four sub-intervals of equal length, approximate the value of $\int_0^{\pi} \sin x\,\mathrm{d}x$.

3. Suppose $f(x) = 2 - 3x$. Using a midpoint Riemann sum with three sub-intervals of equal length, estimate $\int_0^2 (2-3x)\,\mathrm{d}x$. What is the exact value of the definite integral?
   (*Hint: For the second part, do not calculate a limit!*)

4. Consider the quarter circle in the fourth quadrant centered at the origin with a radius of 3. Explain why the integral expression

$$\int_0^3 -\sqrt{9-x^2}\,\mathrm{d}x$$

   gives the *negative* area of the shaded region shown on the right. By using a geometric argument, find the value of the definite integral.
   (*Hint: What is the equation of a circle?*)



5. By using Definition 4.11, the identities provided below it, and the properties of definite integrals, evaluate the following definite integrals.

   (a) $\displaystyle\int_0^4 3\,\mathrm{d}x$ (b) $\displaystyle\int_0^2 (x^2-2x)\,\mathrm{d}x$ (c) $\displaystyle\int_{-1}^3 |x-2|\,\mathrm{d}x$

6. Convert the following limits of Riemann sums into integral expressions. You do not need to evaluate the definite integrals.

   (a) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)\cdot\frac{2}{n}$ (b) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{3n}\tan\left(\frac{\pi i}{3n}\right)$ (c) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{6}{n}\ln\left(6+\frac{18k}{n}\right)$

7. Suppose $\int_{-3}^{10} h(x)\,\mathrm{d}x = 5$, $\int_{10}^{6} h(x)\,\mathrm{d}x = -18$, and $\int_{-3}^{6} k(x)\,\mathrm{d}x = 9$. Evaluate the following integrals:

   (a) $\displaystyle\int_{10}^{-3} 3h(x)\,\mathrm{d}x$ (b) $\displaystyle\int_{-3}^{6} 3(h(x)-4)\,\mathrm{d}x$ (c) $\displaystyle\int_{6}^{-3} (x+k(x))\,\mathrm{d}x$

8. Draw a picture to convince yourself of the following Theorems; don't try to prove them.

   (a) If $f(x) \geq 0$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$.

   (b) If $f(x) \geq g(x)$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$.

   (c) If $m \leq f(x) \leq M$ on $[a,b]$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$.

   (*These Theorems are of particular relevance to our next section.*)

4.2 The Fundamental Theorem of Calculus

As we have seen in the previous section, using limits of Riemann sums to calculate areas is very tedious, and we are very prone to mistakes. Recall that with differentiation, studying the limit definition served the purpose of reinforcing the idea that a derivative represents *instantaneous slope*. Similarly here, our discussion should remind us that definite integrals represent *area under a curve*, and are built using infinite sums of infinitesimally thin pieces.

But of course, in practice, there is absolutely no way we can use the limit method any time we want to evaluate an integral; we need a quicker way to evaluate definite integrals. We will return to this momentarily.

Anti-Differentiation

In many situations in calculus, we know the rate of change of one variable with respect to another, but we do not have a formula which relates the variables. Otherwise said, we have an equation for $f'(x)$, but we would like to have an equation for $f(x)$.

Definition 4.15. The process of finding $f(x)$ from $f'(x)$ is the reverse process of differentiation. We call this *anti-differentiation*.

Example 4.16. Consider the derivative function $f'(x) = x^2$.

From our earlier work with derivatives, we know that, via the power rule, each time we differentiate a power function, the exponent reduces by 1. So it would make sense that $f(x)$ must involve x^3.

If we guess $f(x) = x^3$, then $f'(x) = 3x^2$. So if we start with $f'(x) = \frac{1}{3}x^3$, then $f'(x) = x^2$, as required.

However, also consider $f(x) = \frac{1}{3}x^3 + 5$ or $f(x) = \frac{1}{3}x^3 - 800$ or $f(x) = \frac{1}{3}x^3 + \ln 66$. In all of these cases, we would find the derivative $f'(x) = x^2$.

In fact, there are infinitely many functions of the form $f(x) = \frac{1}{3}x^3 + C$ for some arbitrary constant C that would give $f'(x) = x^2$.

Definition 4.17. If $F(x)$ is a function where $F'(x) = f(x)$, we say:

The *derivative* of $F(x)$ is $f(x)$.

An *anti-derivative* of $f(x)$ is $F(x)$.

Notice the difference in language. A given function has at most one derivative, but (potentially) many anti-derivatives. Indeed, both $\frac{1}{3}x^3$ and $\frac{1}{3}x^3 - 800$ are anti-derivatives of x^2.

Here is the simplest rule for anti-differentiation.

Theorem 4.18 (Reverse Power Rule). If $f(x) = x^n$ where n is constant, then every anti-derivative of f has the form

$$F(x) = \frac{1}{n+1}x^{n+1} + C$$

where C is an arbitrary constant.

The way we think about anti-differentiation is *reversing* differentiation. So, in order to reverse the power rule, we do exactly the opposite of what we did with the power rule. Recall that, with the power rule, we multiply the coefficient by the exponent then reduce the power by 1. To reverse this, we *raise* the power by 1 then *divide* by the new exponent.

Examples 4.19. 1. Find all the anti-derivatives of $g(x) = 3 - 2x$.

Notice that $3 = 3x^0$. So $G(x) = \frac{3}{1}x^{0+1} - \frac{2}{2}x^{1+1} + C = 3x - x^2 + C$

2. Find the anti-derivatives of $f'(x) = \sqrt[3]{x} + \frac{1}{x^2}$.

Like the power rule, the reverse power rule works for non-integer powers too!

$$f'(x) = x^{1/3} + x^{-2} \Longrightarrow f(x) = \frac{1}{4/3}x^{1/3+1} + \frac{1}{-1}x^{-2+1} + C = \frac{3}{4}x^{4/3} - x^{-1} + C$$
$$= \frac{3\sqrt[3]{x^4}}{4} - \frac{1}{x} + C$$

It is standard to give our answer in radical notation if the question does so as well, but this isn't required, of course.

3. For $h(x) = \frac{1}{x} = x^{-1}$, if we try to use the reverse power rule...

$$H(x) = \frac{1}{0}x^{-1+1} + C$$

Clearly we have a problem. We will deal with this later.

Why are we discussing anti-derivatives here? It turns out that the Fundamental Theorem combines:

Anti-Differentiation Find $F(x)$ such that $F'(x) = f(x)$

(Definite) Integration Find the area under the curve $y = f(x)$

It is termed *fundamental* because it provides the link between the two branches of undergraduate/high school calculus: differentiation and integration.

First, we want to think of integrals as functions. We fix the lower limit of a definite integral to be a constant a and let the upper limit be variable. Thus if f is a function defined on an interval containing a and x, then let g be the function[23]

$$g(x) = \int_a^x f(t)\,\mathrm{d}t$$

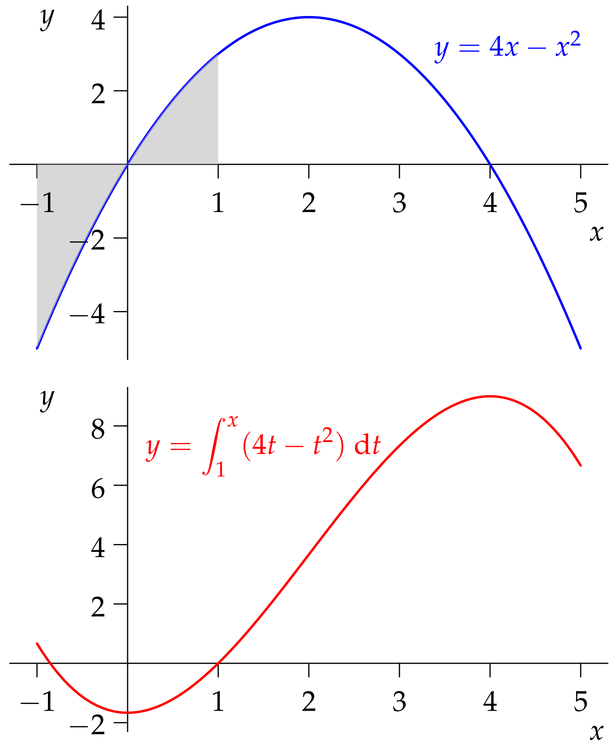
The function g is a function of x.[24]It returns the *net* area under the curve $y = f(x)$ from a up to x. Recall the conventions from the previous sections for how to understand values of definite integrals and net area: in particular, net area is *negative* if either $x < a$ or $f(t) < 0$.

[23]We sometimes call such a function an *accumulation function*.

[24]We use t as our variable in the integrand because, by convention, we don't want the variable in the bounds to be the same as that in the integrand.

Let $g(x) = \int_1^x (4t - t^2)\,\mathrm{d}t$ where $f(x) = 4x - x^2$. Take a look at the following picture. The graph of $y = g(x)$ returns the net area under the curve given by $y = f(x)$ from 1 up to x. Here are some details/points of interest:

- $g(1) = 0$ because of one of the properties of definite integrals; no area!
- $f(x) = 4x - x^2$ has a *relative maximum* when $x = 2$. Interestingly enough, $g(x)$ has a *point of inflection* at $x = 2$ as well.
- On the interval $x < 2$, f is *increasing*, while the graph of g is *concave up*. On the interval $x > 2$, f is *decreasing*, while the graph of g is *concave down*.
- g is *increasing* on the interval $(0, 4)$. On the same interval, $f(x)$ is *positive*.



The relationship between the functions f and g should be clear after these observations. Try sketching the derivative of the lower graph, and you should obtain the upper graph.

Theorem 4.20 (Fundamental Theorem of Calculus, part I). Suppose that f is continuous on the interval $[a, b]$. Then the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous on $[a, b]$, differentiable on (a, b), and its derivative is

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

In essence, the Fundamental Theorem of Calculus (FTC) part I says that if you integrate a function then differentiate, you return to what you started with. Note that a is *any constant*.

Examples 4.21. 1. Find the derivative of $f(x) = \int_3^x \cos(t^2)\,\mathrm{d}t$.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_3^x \cos(t^2)\,\mathrm{d}t = \cos(x^2)$$

2. Switching limits: if the variable bound is at the bottom of the integral, we need to switch the bounds, thus introducing a minus sign, before applying the Theorem, e.g.

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int_\theta^2 e^{\sin t}\,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}\theta}\int_2^\theta e^{\sin t}\,\mathrm{d}t = -e^{\sin\theta}$$

3. FTC part I is really a *derivative rule*. If we roll all the way back to the second chapter, as with all the derivative rules, it can also be combined with the chain rule! So if the variable limit is more complicated, then we must substitute that into t and also *multiply* by its derivative, as per the chain rule. For example,

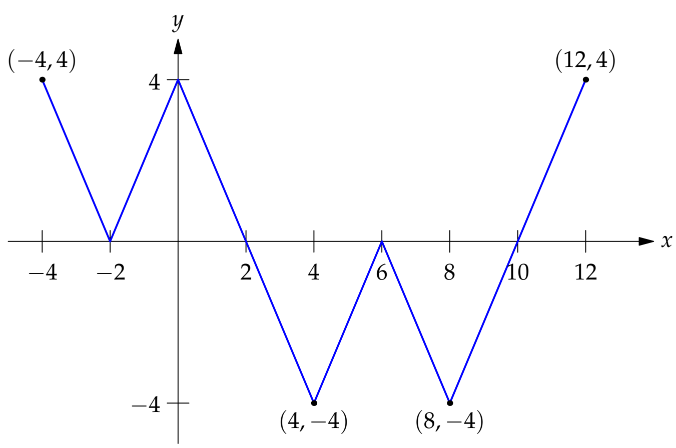
$$F(x) = \int_{-2}^{e^{2x}} (t^3-1)^4 \, \mathrm{d}t \implies F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{-2}^{e^{2x}} (t^3-1)^4 \, \mathrm{d}t = ((e^{2x})^3-1)^4 \cdot 2e^{2x} = 2e^{2x}(e^{6x}-1)^4$$

4. If both the lower and upper limits are variable expressions, then we can separate into two integrals, switch the limits on one integral, and perhaps use the chain rule to finish things off:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \int_{x^2}^{x^3} \sin t \, \mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x} \int_{0}^{x^3} \sin t \, \mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x} \int_{x^2}^{0} \sin t \, \mathrm{d}t && \text{(split into two integrals)} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \int_{0}^{x^3} \sin t \, \mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x} \int_{0}^{x^2} \sin t \, \mathrm{d}t && \text{(switch limits on second integral)} \\
&= 3x^2 \sin(x^3) - 2x \sin(x^2) && \text{(apply FTC with chain rule)}
\end{aligned}$$

The choice of zero as the splitting point in the first line was irrelevant. We could have chosen any constant in the domain of $\sin t$.

5. The Fundamental Theorem can also be used for derivative graph problems! For example, let $g(x) = \int_2^x f(t) \, \mathrm{d}t$. The graph of f is shown below.



To find, for example, $g(0)$, we have

$$g(0) = \int_2^0 f(t) \, \mathrm{d}t = -\int_0^2 f(t) \, \mathrm{d}t = -4 \text{ \{area of triangle\}}$$

Because of FTC part I, we have

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_2^x f(t) \, \mathrm{d}t = f(x)$$

Thus f is simply the derivative of g! So, for instance, on the interval $-2 < x < 2$, $f(x)$ is positive $\implies$ g is increasing. Recall the Theorems from Section 3.1 and 3.2

The second half of the Fundamental Theorem is more widely used. In short, it says that if you differentiate a function and then integrate, you get back (almost) to where you started.

Theorem 4.22 (Fundamental Theorem of Calculus, part II). Suppose that f is continuous on the interval $[a, b]$, and that F is any anti-derivative of f. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Definition 4.23. We have two alternative notations for evaluating the anti-derivative:

$$F(x)\Big|_a^b = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

It is worth taking a step back just to appreciate the beauty of the Theorem. FTC, in essence, is saying that *differentiation* (instantaneous slope) is the reverse operation of *integration* (area under a curve), thus linking the two branches of calculus.

What makes FTC part II interesting is that we can find the area between a graph and the x-axis over an interval *just* by knowing information about the *endpoints* of the anti-derivative on that interval!

This also ties back to our talk in the previous section: we said that the area beneath a derivative graph on an interval represents the *net change* in the value of the original function over that interval. To really spell it out, the net change of F over $[a, b]$ is $F(b) - F(a)$. This is precisely FTC part II!

Note also that the Theorem says *any* anti-derivative of f will do. Thus we may as well choose the simplest anti-derivative: that which has no additional constant.

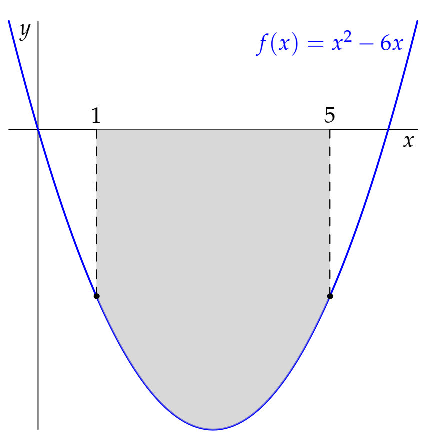
Examples 4.24. 1. Find the area under the graph of $f(x) = x^2 + 1$ from $x = 1$ to $x = 4$.

$$\text{Area} = \int_1^4 (x^2 + 1)\,\mathrm{d}x = \frac{1}{3}x^3 + x\Big|_1^4 = \left(\frac{1}{3}(4)^3 + 4\right) - \left(\frac{1}{3}(1)^3 + 1\right) = 24 \text{ units}^2$$

Exactly the answer we got from the limits method in the previous section, but of course obtained in a much simpler manner by using anti-derivatives.

2. Find the area between the graph of $f(x) = x^2 - 6x$ and the x-axis over the interval $[1, 5]$.

$$\begin{aligned}\text{Area} &= \int_1^5 (x^2 - 6x)\,\mathrm{d}x = \frac{1}{3}x^3 - 3x^2\Big|_1^5 \\ &= \left(\frac{1}{3}(5)^3 - 3(5)^2\right) - \left(\frac{1}{3}(1)^3 - 3(1)^2\right) \\ &= -\frac{92}{3} \text{ units}^2\end{aligned}$$



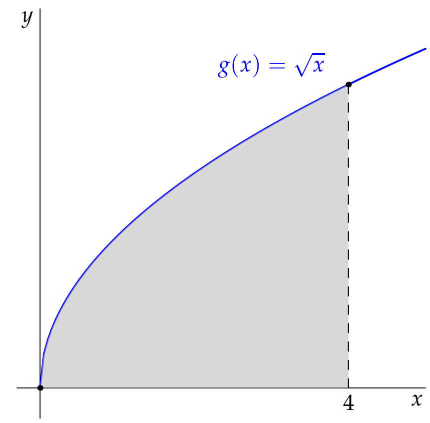
Since the graph of f is below the x-axis on the given interval $[1, 5]$, we should expect that the value of the integral is *negative*. Our answer confirms this.

3. Find the area under the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 4$.

$$\text{Area} = \int_0^4 x^{1/2}\,\mathrm{d}x = \frac{2}{3}x^{3/2}\Big|_0^4$$
$$= \frac{2}{3}\sqrt{4}^3 - \frac{2}{3}\sqrt{0}^3 = \frac{16}{3}\text{ units}^2$$

As a sanity check, since the graph of $\sqrt{x}$ is above the x-axis, the value of the integral should be positive, which it is.

Also always remember that it is advantageous to write roots and such as power functions for easier differentiation or anti-differentiation.



Thus far, we've only considered the anti-derivatives of power functions. We'll explore other methods of anti-differentiation in the next section. We can at least introduce some notation.

Definition 4.25. If f is a function which has an anti-derivative F, then the *indefinite integral* of f is

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

where C is called the *constant of integration*.

From now on, we'll use this notation for anti-derivatives.

Examples 4.26. 1. For $f(x) = x^2 - 3x$, we have

$$\int (x^2 - 3x)\,\mathrm{d}x = \frac{1}{3}x^3 - \frac{3}{2}x^2 + C$$

2. $\displaystyle\int (3x^7 - 19x^3 - 4)\,\mathrm{d}x = \frac{3}{8}x^8 - \frac{19}{4}x^4 - 4x + C$

Be careful: the definite integral $\int_a^b f(x)\,\mathrm{d}x$ is a *number*, while the indefinite integral $\int f(x)\,\mathrm{d}x$ is a *function* or a family of functions: indeed

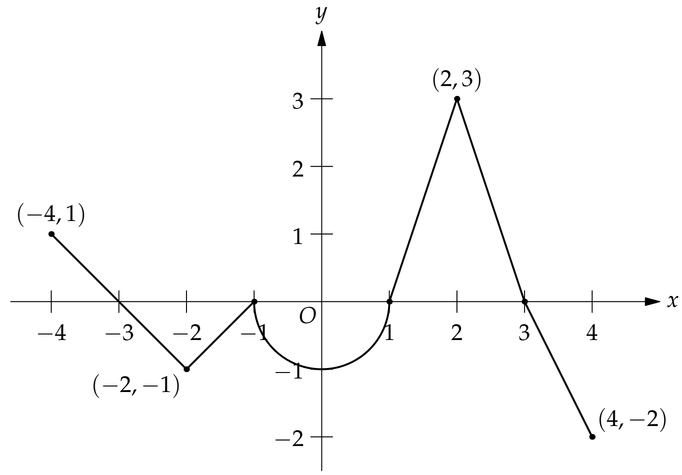
$$\int_a^b f(x)\,\mathrm{d}x = \left[\int f(x)\,\mathrm{d}x\right]_a^b$$

Also as a warning, notice that the statements of the two parts of FTC both insist that f is *continuous*. This is absolutely necessary. For instance, $g(x) = x^{-2}$ is not continuous at $x = 0$. So

$$\int_0^2 x^{-2}\,\mathrm{d}x \neq -x^{-1}\Big|_0^2$$

Integrals like these are called *improper integrals*, and are dealt with in BC.

Exercises 4.2. For the first question, use the graph shown below.



Graph of f

1. The figure above shows the graph of the function f, consisting of five line segments and a semicircle centered at the origin. For $-4 \le x \le 4$, the function g is defined by $g(x) = \int_1^x f(t)\,\mathrm{d}t$.
   (a) Find $g(-4)$ and $g(3)$.
   (b) Does g have a local minimum, local maximum, or neither at $x = -1$? Justify your answer.
   (c) Give the location of any points of inflection of g. Explain your reasoning.
   (d) Find $g''(-3)$ or state that it does not exist. Justify your answer.

2. Evaluate the following, if possible.
   (a) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\displaystyle\int_2^{\theta^3} \cos(3\sqrt{t})\,\mathrm{d}t$
   (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{2x}^{4} \frac{t^2-1}{3-t}\,\mathrm{d}t$
   (c) $\dfrac{\mathrm{d}}{\mathrm{d}z}\displaystyle\int_{2z}^{3\sqrt{z}} e^{\sin u}\,\mathrm{d}u$
   (d) $\displaystyle\int_2^5 \frac{1}{x^3}\,\mathrm{d}x$
   (e) $\displaystyle\int_0^5 2x\,\mathrm{d}x$
   (f) $\displaystyle\int_{-2}^1 (3-x^4)\,\mathrm{d}x$

3. We investigate what happens when we try to apply the Fundamental Theorem to a discontinuous function. Let f be a piecewise-defined function given by

$$f(x) = \begin{cases} 1-x, & x \le 1 \\ x, & x > 1 \end{cases}$$

   (a) Verify that f has a *jump discontinuity* at $x = 1$.
   (b) Find the area between the graph of f and the x-axis on $[0, 2]$, i.e. $\int_0^2 f(x)\,\mathrm{d}x$ by sketching the graph of f and using geometric area formulas.
   (c) Show that $F(x) = \begin{cases} x - \frac{1}{2}x^2, & x < 1 \\ 1 + \frac{1}{2}x^2, & x > 1 \end{cases}$ is an anti-derivative of $f(x)$ whenever $x \ne 1$.
   (d) Show that $\int_0^2 f(x)\,\mathrm{d}x \ne F(2) - F(0)$: the Fundamental Theorem fails.

4.3 Techniques for Integration

Now that we have established a method for finding areas under curves via anti-derivatives, we now need techniques for finding said anti-derivatives.

This is not always possible, unfortunately. If we are given some function, it is often very straightforward to find its derivative. The reverse process is *much* more challenging. The way we should frame our thinking in this section is to try to *reverse differentiation*. Given that, you may need to review many of the derivative rules; they are important here.

First, we will take a few of the properties of derivatives to aid us here.

Theorem 4.27 (Linearity). If f, g are functions and a, b are constants, then

$$\int (af(x) \pm bg(x))\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x \pm b\int g(x)\,\mathrm{d}x$$

Like with differentiation, this Theorem says that constant coefficients may be ignored or factored out in integration, and that sums and differences of functions can be integrated separately.

Common Functions

If you are familiar with the derivative rules, the following anti-derivatives should be very obvious.

Theorem 4.28. Let n be constant, where $n \neq 1$. Then the following are true:

$$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C \qquad \int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C \qquad \int e^x\,\mathrm{d}x = e^x + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C \qquad \int \cos x\,\mathrm{d}x = \sin x + C \qquad \int \sec^2 x\,\mathrm{d}x = \tan x + C$$

$$\int \sec x \tan x\,\mathrm{d}x = \sec x + C \qquad \int \csc^2 x\,\mathrm{d}x = -\cot x + C \qquad \int \csc x \cot x\,\mathrm{d}x = -\csc x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C \qquad \int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C \qquad \int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{arcsec}|x| + C$$

For $\int x^{-1}\,\mathrm{d}x$, the reason we need absolute value is because for positive x, $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$, and for negative x, $\frac{\mathrm{d}}{\mathrm{d}x}\ln(-x) = \frac{1}{-x} \cdot -1 = \frac{1}{x}$. The same reasoning goes for inverse secant.

Examples 4.29. 1. $\displaystyle\int \left(3 + \frac{2}{x}\right)\mathrm{d}x = 3x + 2\ln|x| + C$

2. $\displaystyle\int (3\sec^2 x - 2e^x)\,\mathrm{d}x = 3\tan x - \frac{2}{3}e^x + C$

3. For $g(x) = \cos x + \dfrac{x^3 + \sqrt{x}}{x^2}$, the key is to do some algebraic manipulation so that the terms are listed in Theorem 4.28. Here, we divide first:

$$g(x) = \cos x + \frac{x^3}{x^2} + \frac{x^{1/2}}{x^2} = \cos x + x + x^{-3/2} \implies \int g(x)\,\mathrm{d}x = \sin x + \frac{1}{2}x^2 - 2x^{-1/2} + C$$

Of course, you can check your answer by differentiating and making sure it matches the integrand.

There are not many tips here other than to memorize the anti-derivatives listed in 4.28. Once you are very familiar with them, you will find yourself algebraically fiddling around until you get to the expressions above.

The Substitution Rule

This is the emphasis of integration techniques. We can easily differentiate simple expressions, but for something like

$$\int x^2 \sin(x^3)\,\mathrm{d}x$$

the anti-derivative is not easily recognizable. If it is possible for us to integrate this, there are several possible approaches:

Guess and check: Make a guess, differentiate it to check, and modify your guess until you get it right! Over time, you will find yourself using this method more and more. It is definitely faster than the substitution rule, but requires some intuition, which comes with practice.

Use a rule: Recalling differential calculus, we might try to formulate some helpful rules based on the chain and product rules. The product rule will be resurrected later in BC (as *integration by parts*). The substitution rule explains how we may *reverse the chain rule*.

In essence, our goal is to reduce an integral to one listed in 4.28.

Theorem 4.30 (Substitution Rule[25]). If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Proof. Let $u = g(x)$, and let F be an anti-derivative of f on the interval I, so that $F'(u) = f(u)$. The chain rule says that

$$\frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

Any expression for a derivative may be rephrased using indefinite integrals. Thus,

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) = F(u) = \int f(u)\,\mathrm{d}u$$

as required. ∎

Here is the method:

1. Identify the 'most complicated part' of the integrand. Typically, we set u equal to whatever is 'inside' the most complicated part.

[25] Also called the 'u-substitution' method.

2. Differentiate u with respect to x, then solve for $\mathrm{d}x$.

3. Replace the original integrand with the expressions obtained in the first two steps until everything is in terms of u. By this point, the integral should match one listed in 4.28.

4. Integrate then substitute back the expression for u to get your answer in terms of x.

The suggestion in the first step is merely a guideline. Sometimes, you will have to be clever with your substitution. Other times, the question may supply us with an expression for u. If you happen to make an incorrect choice of u, go back and try another!

Examples 4.31. 1. Find $\int (2x-1)(x^2-x)^2 \,\mathrm{d}x$.

The 'most complicated part' seems to be $(x^2-x)^2$. So we set $u = x^2 - x$. From there,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 1 \implies \mathrm{d}x = \frac{1}{2x-1}\,\mathrm{d}u$$

We can now replace the pieces in the original integral, integrate and substitute back at the end:

$$\begin{aligned}\int (2x-1)(x^2-x)^2 \,\mathrm{d}x &= \int (2x-1)u^2 \cdot \frac{1}{2x-1}\,\mathrm{d}u \\ &= \int u^2 \,\mathrm{d}u = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2-x)^3 + C\end{aligned}$$

After making the substitutions, we were confident that we had done everything correctly because every x term was replaced. We can also check our answer by differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}(x^2-x)^3 + C\right) = (x^2-x)^2 \cdot (2x-1)$$

2. Find $\int x^2 \sin(x^3) \,\mathrm{d}x$.

We seek out the 'most complicated part', which seems to be $\sin(x^3)$, and set $u = x^3$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 \implies \mathrm{d}x = \frac{1}{3x^2}\,\mathrm{d}u$$

Now replace and integrate:

$$\int x^2 \sin(x^3) \,\mathrm{d}x = \int x^2 \sin(u) \cdot \frac{1}{3x^2}\,\mathrm{d}u = \frac{1}{3}\int \sin u \,\mathrm{d}u = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos x^3 + C$$

3. Find $\int 2e^{5x} \,\mathrm{d}x$.

Since there seems only to be one 'part', we set $u = 5x$:

$$\begin{aligned}&\frac{\mathrm{d}u}{\mathrm{d}x} = 5 \implies \mathrm{d}x = \frac{1}{5}\,\mathrm{d}u \\ &\implies \int 2e^{5x} \,\mathrm{d}x = \int 2e^u \cdot \frac{1}{5}\,\mathrm{d}u = \frac{2}{5}\int e^u \,\mathrm{d}u = \frac{2}{5}e^u + C = \frac{2}{5}e^{5x} + C\end{aligned}$$

4. Evaluate $\int \frac{1}{4x+3}\,\mathrm{d}x$.

   Between the numerator and denominator, the 'most complicated part' seems just to be the denominator. There is nothing 'inside', so we just set $u = 4x + 3$:

   $$\frac{\mathrm{d}u}{\mathrm{d}x} = 4 \Longrightarrow \mathrm{d}x = \frac{1}{4}\,\mathrm{d}u$$
   $$\Longrightarrow \int \frac{1}{4x+3}\,\mathrm{d}x = \int \frac{1}{u}\cdot\frac{1}{4}\,\mathrm{d}u = \frac{1}{4}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|4x+3| + C$$

5. Compute $\int \sec^2(7x)\,\mathrm{d}x$.

   As we do more examples, it becomes more obvious what the anti-derivative should look like. We guess $\tan(7x)$:

   $$\frac{\mathrm{d}}{\mathrm{d}x}\tan(7x) = 7\sec^2(7x)$$

   So we include a $\frac{1}{7}$ to balance. Thus

   $$\int \sec^2(7x)\,\mathrm{d}x = \frac{1}{7}\tan(7x) + C$$

Substitutions in Definite Integrals

The substitution rule can be applied directly to definite integrals. The important point is that you must *change the limits*!

Theorem 4.32. If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

Examples 4.33. 1. To evaluate $\int_0^4 \sqrt{2x-1}\,\mathrm{d}x$, we substitute $u = 2x + 1$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \Longrightarrow \mathrm{d}x = \frac{1}{2}\,\mathrm{d}u, \qquad u(0) = 2\cdot 0 + 1 = 1, \quad u(4) = 2\cdot 4 + 1 = 9$$

It follows that

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\,\mathrm{d}x &= \int_1^9 \sqrt{u}\cdot\frac{1}{2}\,\mathrm{d}u && \text{(substitute \textit{and} change limits)}\\ &= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^9 && \text{(find anti-derivative)}\\ &= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{26}{3} && \text{(evaluate and simplify)}\end{aligned}$$

Notice that once we substitute and change the limits, we never see x again.

An alternative to changing the limits is to first compute the *indefinite integral* then substitute back. For example:

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

Therefore

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{3/2}\bigg|_0^4 = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{26}{3}$$

Both of these methods are acceptable. What is incorrect is to *mix* them.

2. Compute $\int_{3\pi/2}^{2\pi} e^{\cos x}\sin x\,dx$.

   The ugliest expression is $e^{\cos x}$, so we set $u = \cos x$, then

$$\frac{du}{dx} = -\sin x \Longrightarrow dx = \frac{-1}{\sin x}\,du, \qquad u\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} = 0, \quad u(2\pi) = \cos 2\pi = 1$$

   Therefore

$$\int_{3\pi/2}^{2\pi} e^{\cos x}\sin x\,dx = \int_0^1 e^u \sin x\cdot\frac{-1}{\sin x}\,du = -\int_0^1 e^u\,du = -e^u\bigg|_0^1 = -(e-1) = 1-e$$

Inverse Trigonometric Functions

Recognizing that an integrand *looks* like the derivative of a function we studied in a previous chapter is a helpful skill.

Examples 4.34. 1. To find $\int \frac{1}{\sqrt{1-9x^2}}\,dx$, perhaps we try the substitution $u = 1-9x^2$:

$$\frac{du}{dx} = -18x \Longrightarrow dx = \frac{-1}{18x}\,du \Longrightarrow \int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{u}}\cdot\frac{-1}{18x}\,du$$

This substitution is no good, since we haven't removed all the x-terms; we need to try another. We notice that the integrand looks like the derivative of inverse sine. After some algebra:

$$\int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

Now we try the substitution $u = 3x$:

$$\frac{du}{dx} = 3 \Longrightarrow dx = \frac{1}{3}\,du$$

Thus

$$\int \frac{1}{\sqrt{1-(3x)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\cdot\frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\arcsin u + C = \frac{1}{3}\arcsin 3x + C$$

2. When we recognize that an integrand is the derivative of an inverse trigonometric function, we want the constant in the denominator to be 1. Consider $\int \frac{1}{4+x^2}\,dx$:

$$\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4(1+\frac{1}{4}x^2)}\,dx = \frac{1}{4}\int \frac{1}{1+(\frac{1}{2}x)^2}\,dx$$

Hence we choose the substitution $u = \frac{1}{2}x$:

$$\frac{du}{dx} = \frac{1}{2} \Longrightarrow dx = 2\,du$$

So we have

$$\frac{1}{4}\int \frac{1}{1+(\frac{1}{2}x)^2}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin\frac{x}{2} + C$$

3. Evaluate $\int \frac{10x}{x^2\sqrt{x^4-16}}\,dx$.

$$\int \frac{10x}{x^2\sqrt{x^4-16}}\,dx = 10\int \frac{x}{x^2\sqrt{16(\frac{1}{16}x^4-1)}}\,dx = \frac{10}{4}\int \frac{x}{x^2\sqrt{(\frac{1}{4}x^2)^2-1}}\,dx$$

hence we choose the substitution $u = \frac{1}{4}x^2$ and $4u = x^2 \Longrightarrow dx = \frac{1}{2x}\,du$:

$$\frac{5}{2}\int \frac{x}{4u\sqrt{u^2-1}}\cdot\frac{1}{2x}\,du = \frac{5}{16}\int \frac{1}{u\sqrt{u^2-1}}\,du = \frac{5}{16}\operatorname{arcsec} u + C = \frac{5}{16}\operatorname{arcsec}\frac{x^2}{4} + C$$

We can remove the absolute value since $\frac{x^2}{4}$ is always positive. For complicated questions like these, it may be worth the time to differentiate to check our answer:

$$\frac{d}{dx}\left(\frac{5}{16}\operatorname{arcsec}\frac{x^2}{4} + C\right) = \frac{5}{16}\cdot\frac{2x}{\frac{x^2}{4}\sqrt{\frac{x^4}{16}-1}} = \frac{10x}{x^2\sqrt{x^4-16}}$$

So far, it seems functions which are fractions are the hardest to integrate. Another tip is to look at the degree of the numerator and denominator. If the degree of the numerator is greater than or equal to that of the denominator, we should do long division first. If you remember how to perform synthetic division, that would also work with a linear divisor.

Examples 4.35. 1. We want to find $\int \frac{8x-14}{2x^2-7x+7}\,dx$. Since the numerator has lower degree than the denominator, we may integrate without dividing first. Of the two, the denominator is uglier, so we let $u = 2x^2-7x+7$:

$$\frac{du}{dx} = 4x-7 \Longrightarrow dx = \frac{1}{4x-7}\,du$$

Therefore

$$\int \frac{8x-14}{2x^2-7x+7}\,dx = \int \frac{8x-14}{u}\cdot\frac{1}{4x-7}\,du = 2\int \frac{1}{u}\,du = 2\ln|u| + C$$
$$= 2\ln|2x^2-7x+7| + C$$

2. For $\int \frac{5x^2+2x-1}{x^2+1}\,\mathrm{d}x$, the numerator has same degree as the denominator, so we divide first:

$$\begin{array}{r|l} & \quad\quad\quad\quad 5 \\ x^2+1) & 5x^2+2x-1 \\ & -5x^2 \quad\quad -5 \\ \hline & \quad\quad 2x-6 \end{array}$$

Be sure to review long division if you've forgotten how to do it.

The quotient is 5 with remainder $2x-6$. So we can rewrite the integral as

$$\int \frac{5x^2+2x-1}{x^2+1}\,\mathrm{d}x = \int \left(5+\frac{2x-6}{x^2+1}\right)\mathrm{d}x$$

We can separate the integrand even further:

$$\int \left(5+\frac{2x-6}{x^2+1}\right)\mathrm{d}x = \int 5\,\mathrm{d}x + \int \frac{2x}{x^2+1}\,\mathrm{d}x + \int \frac{-6}{x^2+1}\,\mathrm{d}x$$

The first integral is simple. For the second, we let $u = x^2+1 \implies \mathrm{d}u = 2x\,\mathrm{d}x$, so

$$\int \frac{2x}{x^2+1}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln(x^2+1) + C$$

For the last, the anti-derivative is plainly inverse tangent:

$$-6\int \frac{1}{x^2+1}\,\mathrm{d}x = -6\arctan x + C$$

Putting together, our final answer is

$$\int \frac{5x^2+2x-1}{x^2+1}\,\mathrm{d}x = 5x + \ln(x^2+1) - 6\arctan x + C$$

Another niche tip is that if the degree of the denominator is at least two greater than that of the numerator, the anti-derivative is likely arctangent. Otherwise, the anti-derivative is likely natural logarithm.

Completing the Square

Recall the method of *completing the square*: creating a *perfect square trinomial*[26]by manipulating the constant of a trinomial. For instance,

$$x^2+10x+17 = (x^2+10x+25) - 8 = (x+5)^2 - 8$$

where -8 is used to balance. If needed, the constant is $(\frac{b}{2})^2$.

Sometimes we may need to complete the square to change a denominator so that we get an integrand that looks like the derivative of an inverse tangent function.

[26]A trinomial that can be expressed as the square of a binomial.

Examples 4.36. 1. Find $\int \frac{1}{x^2+2x+2}\,\mathrm{d}x$. We complete the square in the denominator:

$$\int \frac{1}{x^2+2x+2}\,\mathrm{d}x = \int \frac{1}{(x^2+2x+1)+1}\,\mathrm{d}x = \int \frac{1}{(x+1)^2+1}\,\mathrm{d}x$$

So we let $u = x+1 \implies \mathrm{d}u = \mathrm{d}x$. So

$$\int \frac{1}{(x+1)^2+1}\,\mathrm{d}x = \int \frac{1}{u^2+1}\,\mathrm{d}u = \arctan u + C = \arctan(x+1) + C$$

2. Find $\int \frac{\mathrm{d}x}{x^2+6x+25}$. Once again, we complete the square:

$$\int \frac{\mathrm{d}x}{x^2+6x+25} = \int \frac{\mathrm{d}x}{(x^2+6x+9)+16} = \int \frac{\mathrm{d}x}{(x+3)^2+16}$$

We then do the usual algebra after recognizing the derivative of inverse tangent:

$$\int \frac{\mathrm{d}x}{(x+3)^2+16} = \int \frac{\mathrm{d}x}{16(\frac{1}{16}(x+3)^2+1)} = \frac{1}{16}\int \frac{\mathrm{d}x}{(\frac{x+3}{4})^2+1}$$

Let $u = \frac{x+3}{4} \implies \mathrm{d}x = 4\,\mathrm{d}u$. Then substitute and integrate:

$$\frac{1}{16}\int \frac{\mathrm{d}x}{(\frac{x+3}{4})^2+1} = \frac{1}{16}\int \frac{1}{u^2+1}\cdot 4\,\mathrm{d}u = \frac{1}{4}\arctan u + C = \frac{1}{4}\arctan\left(\frac{x+3}{4}\right) + C$$

Powers of Sine and Cosine

For an integral such as

$$\int \sin^2 x\,\mathrm{d}x,$$

a simple substitution $u = \sin x$ is insufficient.
Recall the *double angle formulas* for cosine:

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\implies \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta, \quad \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

Also recall the *Pythagorean identity*:

$$\sin^2\theta + \cos^2\theta = 1$$

These trigonometric identities can be used to help us integrate sine and cosine functions raised to some power. Here are the guidelines:

If the power is *even*, use a cosine double angle identity then integrate.

If the power is *odd*, extract a cos or sin term then use the Pythagorean identity to create an integrand that can be dealt with using u-substitution.

Repeated iterations of these steps may be required if the power is higher.

Examples 4.37. 1. We find $\int \sin^2 x \, \mathrm{d}x$. Since the power is even, we use the cosine double angle formula involving sine:

$$\int \sin^2 x \, \mathrm{d}x = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) \mathrm{d}x = \frac{1}{2}x - \frac{1}{4} \sin 2x + C$$

Hopefully we are able to anti-differentiate simpler expressions such as above via guessing now.

2. For $\int \cos^3 4x \, \mathrm{d}x$, since the power is odd, we extract a $\cos 4x$ term then use the identity:

$$\begin{aligned} \int \cos^3 4x \, \mathrm{d}x &= \int \cos 4x \cdot \cos^2 4x \, \mathrm{d}x = \int \cos 4x \left(1 - \sin^2 4x\right) \mathrm{d}x \\ &= \int \cos 4x \, \mathrm{d}x - \int \cos 4x \left(\sin 4x\right)^2 \, \mathrm{d}x \end{aligned}$$

The first integral is trivial. For the second, we let $u = \sin 4x \implies \mathrm{d}x = \frac{1}{4\cos 4x} \, \mathrm{d}u$. So

$$\begin{aligned} - \int \cos 4x \left(\sin 4x\right)^2 \, \mathrm{d}x &= - \int \cos 4x \cdot u^2 \cdot \frac{1}{4 \cos 4x} \, \mathrm{d}u = -\frac{1}{4} \int u^2 \, \mathrm{d}u \\ &= -\frac{1}{4} \cdot \frac{1}{3} u^3 + C = -\frac{1}{12} \sin^3 4x + C \end{aligned}$$

Therefore

$$\int \cos^3 4x \, \mathrm{d}x = \frac{1}{4} \sin 4x - \frac{1}{12} \sin^3 4x + C$$

3. For $\int \cos^4 5x \, \mathrm{d}x$, we use the double angle identity *twice*:

$$\begin{aligned} \int \cos^4 5x \, \mathrm{d}x &= \int \cos^2 5x \cdot \cos^2 5x \, \mathrm{d}x = \int \left(\frac{1}{2} + \frac{1}{2} \cos 10x \right) \left(\frac{1}{2} + \frac{1}{2} \cos 10x \right) \mathrm{d}x \\ &= \int \left(\frac{1}{4} + \frac{1}{2} \cos 10x + \frac{1}{4} \cos^2 10x \right) \mathrm{d}x \end{aligned}$$

The first and second integrals are simple. For the third, we would need to, once again, use a cosine double angle formula:

$$\frac{1}{4} \int \cos^2 10x \, \mathrm{d}x = \frac{1}{4} \int \left(\frac{1}{2} + \frac{1}{2} \cos 20x \right) \mathrm{d}x = \frac{1}{8}x + \frac{1}{160} \sin 20x + C$$

Combining, our final answer is

$$\int \cos^4 5x \, \mathrm{d}x = \frac{1}{4}x + \frac{1}{20} \sin 10x + \frac{1}{8}x + \frac{1}{160} \sin 20x + C = \frac{3}{8}x + \frac{1}{20} \sin 10x + \frac{1}{160} \sin 20x + C$$

This section is quite heavy on the examples, but there is no better way to master integration than repeated practice! There are, of course, some techniques that haven't been covered in detail here, but being able to compute integrals also requires some ingenuity.

Exercises 4.3. 1. Evaluate the following indefinite integrals:

(a) $\int 4x^3(2+x^4)^3\,dx$ (b) $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$ (c) $\int \frac{4z-10}{5z-z^2}\,dz$

(d) $\int \frac{\sin x}{\sqrt{\cos x}}\,dx$ (e) $\int \sec^2\theta\tan^2\theta\,d\theta$ (f) $\int \frac{4}{x\ln x}\,dx$

(g) $\int \sin^2\frac{x}{2}\,dx$ (h) $\int \cos^3 7x\,dx$ (i) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

(j) $\int \frac{3}{x^2-4x+13}\,dx$ (k) $\int \frac{1}{25+9x^2}\,dx$ (l) $\int \csc 3\theta\cot 3\theta\,d\theta$

2. Use the substitution $u=\ln(x^2+7)$ to evaluate $\int \frac{x\ln(x^2+7)}{x^2+7}\,dx$.

3. (a) Find the derivative of $y=\ln(\cos x)$.
   (b) Hence find $\int \tan x\,dx$.

4. (a) Find the derivative of $y=2^x$.
   (b) Hence find $\int 2^x\,dx$.

5. Compute the following definite integrals:

(a) $\int_1^2 \frac{x}{(2+x^2)^2}\,dx$ (b) $\int_0^{\pi/6} \sin^2\theta\cos\theta\,d\theta$ (c) $\int_0^3 x\sqrt{x^2+16}\,dx$

6. Selected values for differentiable functions f, g are shown in the table below:

x	-3	-1	2	4
$f(x)$	2	-5	3	0
$g(x)$	-1	6	-8	2

Compute $\int_{-1}^{2} f'(g(x))g'(x)\,dx$.

(Hint: Find a suitable substitution...)

7. (Hard) Consider the right triangle depicted at the right.

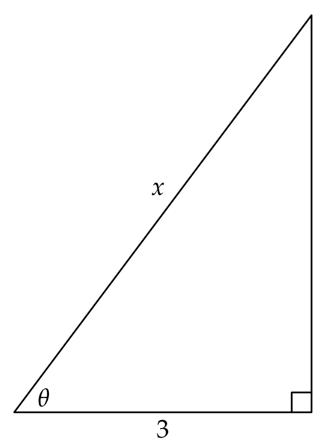
   (a) Find the length of the leg opposite angle θ.

   (b) Plainly $x=3\sec\theta$. Differentiate x with respect to θ.

   (c) Use the substitution $x=3\sec\theta$ and the triangle on the right to compute the indefinite integral

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

(Hint: Recall the identity $1+\tan^2\theta=\sec^2\theta$.*)*



5 Applications of Integration

5.1 Areas and Average Value

We know that the signed area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$ is

$$\text{Area} = \int_a^b f(x)\,\mathrm{d}x$$

Of course, if a problem isn't nice enough to provide us with an interval, and we asked to find the area of a region *bounded* by a curve and the x-axis, we would have to find the limits of the interval on our own. Recall that, to find the x-intercepts of a function, we set $f(x) = 0$.

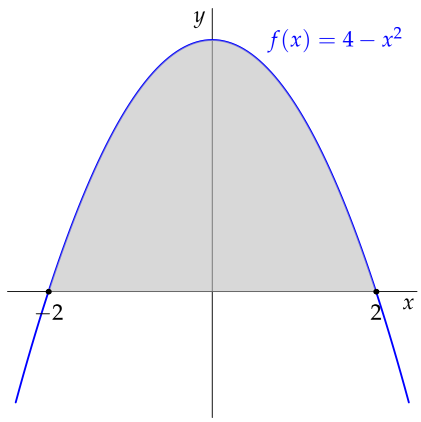
Examples 5.1. 1. Find the area of the region bounded by the graph of $f(x) = 4 - x^2$ and the x-axis.

To find the bounds of the integral, we find the x-intercepts:

$$4 - x^2 = 0 \implies x^2 = 4 \implies x = \pm 2$$

Therefore the area of the bounded region is

$$\begin{aligned}\text{Area} &= \int_{-2}^{2} (4 - x^2)\,\mathrm{d}x = 4x - \frac{1}{3}x^3\Big|_{-2}^{2} \\ &= \left(4(2) - \frac{1}{3}(2)^3\right) - \left(4(-2) - \frac{1}{3}(-2)^3\right) \\ &= \frac{32}{3}\ \text{units}^2\end{aligned}$$



2. Find the area of the region in the first quadrant bounded by $y = \sqrt{4 - x}$ and the x-axis.

Since the problem specifies first quadrant, the lower bound is plainly $x = 0$. To find the upper bound, find the x-intercept:

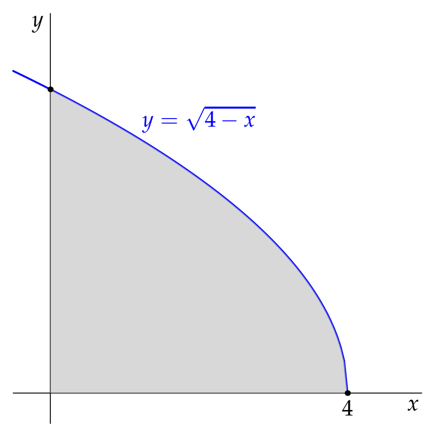
$$\sqrt{4 - x} = 0 \implies 4 - x = 0 \implies x = 4$$

Therefore the area of the region is

$$\int_0^4 \sqrt{4 - x}\,\mathrm{d}x$$

We use substitution $u = 4 - x \implies \mathrm{d}u = -\mathrm{d}x$ and $u(0) = 4$, $u(4) = 0$:

$$\begin{aligned}\text{Area} &= \int_0^4 \sqrt{4 - x}\,\mathrm{d}x = -\int_4^0 \sqrt{u}\,\mathrm{d}u = \int_0^4 u^{1/2}\,\mathrm{d}u \\ &= \frac{2}{3}u^{3/2}\Big|_0^4 = \frac{2}{3}(4)^{3/2} - 0 = \frac{16}{3}\ \text{units}^2\end{aligned}$$



Moreover, when we ask for area, what we really want is *total area*, ignoring the negatives that come with regions below the x-axis.

Definition 5.2. The *total area* between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$ is

$$\text{Total Area} = \int_a^b |f(x)| \, dx$$

Be careful: the total area is, in general, *not* the absolute value of the net area. That is, not

$$\left| \int_a^b f(x) \, dx \right|$$

The upshot is that we need to find the intervals for which the graph of the function is above or below the x-axis *separately*, and flip signs accordingly.

Theorem 5.3. Suppose that $f(x) \geq 0$ on $[a, b]$ and $g(x) \leq 0$ on $[c, d]$. Then

$$\int_a^b |f(x)| \, dx = \int_a^b f(x) \, dx \quad \text{and} \quad \int_c^d |g(x)| \, dx = -\int_c^d g(x) \, dx$$

Remember that absolute value flips parts of the graph of a function below the x-axis above and leaves parts above the x-axis untouched. So the minus should cancel out with the negative that comes from being below the x-axis.

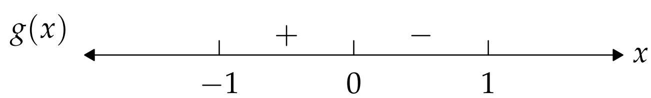
Examples 5.4. 1. Find the total area enclosed by the graph of $g(x) = x^3 - x$ and the x-axis.

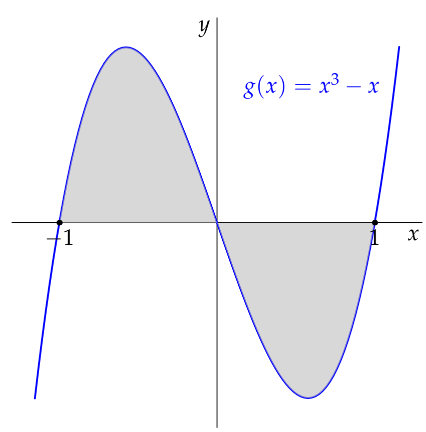
To find the bounds, we get the intercepts:

$$x - x^3 = x(1 - x^2) = x(1 - x)(1 + x) = 0 \implies x = 0, \pm 1$$

For $[-1, 0]$, $g(x) \geq 0$ and for $[0, 1]$, $g(x) \leq 0$. If you're unsure, you can sketch a graph. And if you're unable to draw a graph, you can also use a sign chart; the continuity of our functions guarantees that the signs can only change at their zeros.

$$g(-0.5) > 0 \qquad g(0.5) < 0$$





Thus the total area of the bounded region is

$$\begin{aligned} \text{Area} &= \int_{-1}^0 (x^3 - x) \, dx - \int_0^1 (x^3 - x) \, dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 \right]_{-1}^0 - \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 \right]_0^1 = \frac{1}{2} \text{ units}^2 \end{aligned}$$

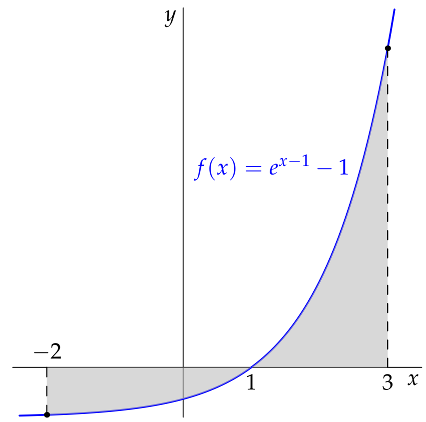
2. Find the area of the region bounded by $f(x) = e^{x-1} - 1$ between $x = -2$ and $x = 3$.

We are already given the interval $[-2, 3]$ to find area, but we need to check whether the graph of f crosses the x-axis at any value between:

$$e^{x-1} - 1 = 0 \implies e^{x-1} = 1 \implies x - 1 = 0 \implies x = 1$$

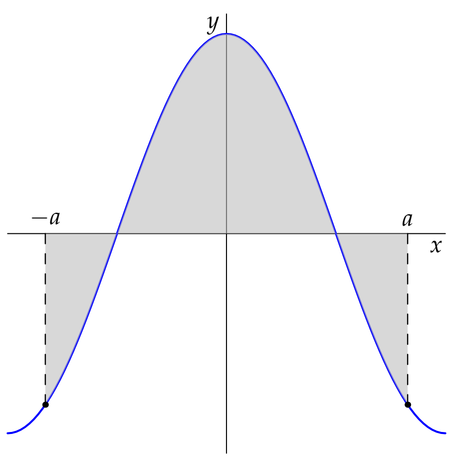
On $[-2, 1]$, $f(x) \leq 0$ and on $[1, 3]$, $f(x) \geq 0$. Again, sketch a graph or test values if you're unsure. The total area is thus

$$\begin{aligned}
\text{Area} &= -\int_{-2}^{1} (e^{x-1} - 1)\,\mathrm{d}x + \int_{1}^{3} (e^{x-1} - 1)\,\mathrm{d}x \\
&= -\left[e^{x-1} - x\right]_{-2}^{1} + \left[e^{x-1} - x\right]_{1}^{3} \\
&= -\big((1-1) - (e^{-3} + 2)\big) + \big((e^2 - 3) - (1 - 1)\big) \\
&= e^{-3} + e^2 - 1 \text{ units}^2
\end{aligned}$$

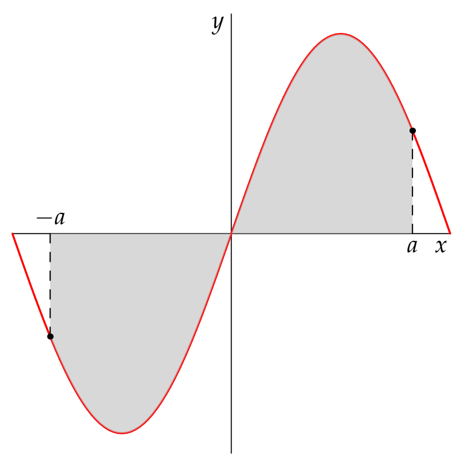


Symmetry

We can use *symmetry* of graphs of functions to our advantage when finding area. Recall the notion of *even* and *odd functions*: a function f is even if $f(x) = f(-x)$ and odd if $f(-x) = -f(x)$. Otherwise said, f is even if its graph is *symmetric about the y-axis*, and odd if its graph has *rotational symmetry about the origin*.



An even function



An odd function

Functions which are even or odd are fairly straightforward, if integrated over a *symmetric interval* $[-a, a]$. This can save us a slight bit of calculation.

Theorem 5.5. Suppose that f is continuous on $[-a, a]$.

If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

Moreover, if we want *total area*, then if f is either even *or* odd:

$$\int_{a}^{a} |f(x)|\,dx = 2\int_{0}^{a} |f(x)|\,dx$$

Using this Theorem is not strictly necessary, but computing with 0 is often quite easy.

Example 5.1 cont. 1. The alternative way to find the area of the region bounded by the graph of $f(x) = 4 - x^2$ and the x-axis is to realize that the graph of f is symmetric about the y-axis, i.e. f is even. Therefore

$$\text{Area} = 2\int_{0}^{2} (4 - x^2)\,dx = 2\left[4x - \frac{1}{3}x^3\right]_0^2 = 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}\ \text{units}^2$$

Areas Between Curves

Our current methods allow us to find area between the graph of a function and the x-axis, but if we need the area *between* two curves, then we add an extra step.

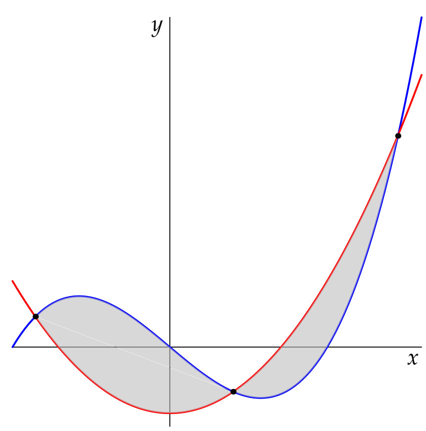
Theorem 5.6. If $f(x) \geq g(x)$ on $[a, b]$, then the area between the graphs of f and g over $[a, b]$ is the difference between the areas under f and g:

$$\text{Area} = \int_{a}^{b} f(x)\,dx - \int_{a}^{b} g(x)\,dx = \int_{a}^{b} (f(x) - g(x))\,dx$$

The common advice is 'top minus bottom'. And of course, there may be multiple regions bounded by the graphs of f and g, meaning whichever function is above or below may change.

The process is the same as before. Find the x-values of the *intersections* of the curves by setting the equations of the curves equal to one another (essentially solving a *system*) in order to find the x-bounds.

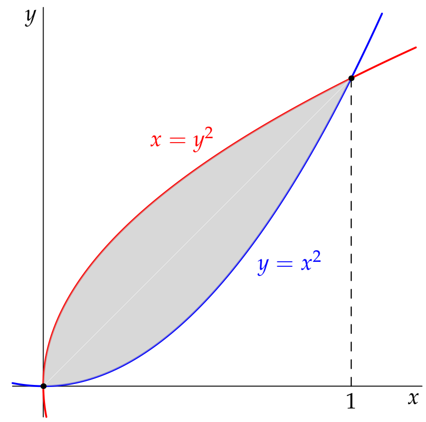
This method works regardless of whether the graphs of f and g are above or below the x-axis.



Examples 5.7. 1. Find the area of the region bounded by the curves $y = x^2$ and $x = y^2$.

First, since we are integrating with respect to x, we should get the equations of our curves as functions of x. Namely, the second curve can be rewritten as $y = \pm x^{1/2}$. If you recognize that the graph of $y^2 = x$ is a right-opening parabola, we can ignore the negative square root. The intersection is

$$x^{1/2} = x^2 \implies x = x^4$$
$$\implies x - x^4 = x(1 - x^3) = 0 \implies x = 0, 1$$



We can sketch a graph or test a point to see that $x^{1/2} \geq x^2$ on $[0, 1]$, so the 'top' function is $x^{1/2}$ and 'bottom' is x^2:

$$\text{Area} = \int_0^1 (x^{1/2} - x^2)\,\mathrm{d}x = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3\bigg|_0^1 = \frac{1}{3}\ \text{units}^2$$

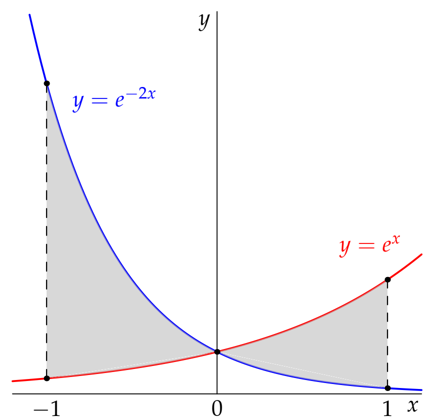
2. Find the total area between the curves $y = e^x$ and $y = e^{-2x}$ between $x = -1$ and $x = 1$.

We first find where the curves meet by equating:

$$e^x = e^{-2x} \implies x = -2x \implies x = 0$$

On $[-1, 0]$, $e^{-2x} \geq e^x$, and on $[0, 1]$, $e^{-3x} \leq e^x$. Therefore

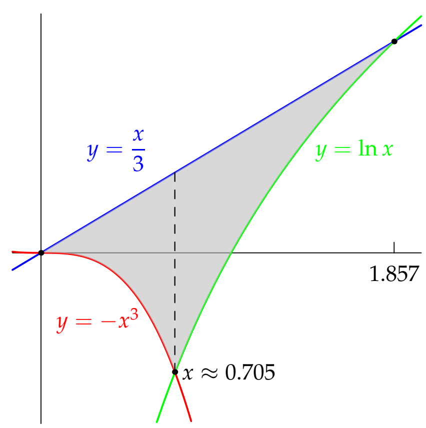
$$\text{Area} = \int_{-1}^{0} (e^{-2x} - e^x)\,\mathrm{d}x + \int_0^1 (e^x - e^{-2x})\,\mathrm{d}x$$
$$= \left[-\frac{1}{2}e^{-2x} - e^x\right]_{-1}^{0} + \left[e^x + \frac{1}{2}e^{-2x}\right]_0^1$$
$$= e + e^{-1} + \frac{1}{2}(e^2 + e^{-2}) - 3\ \text{units}^2$$



3. (Calculator) Find the area of the region bounded by the curves $y = \ln x$, $3y = x$, and $y + x^3 = 0$.

First, after isolating y and getting $y = \frac{x}{3}$ and $y = -x^3$, if we graph using our calculators, we note that the 'bottom' function *changes*; we can find the intersections with the 'intersect' feature:

$$-x^3 = \frac{x}{3} \implies x = 0, \quad -x^3 = \ln x \implies x \approx 0.705 = A$$
$$\frac{x}{3} = \ln x \implies x \approx 1.857 = B$$



Therefore the area of the enclosed region is given by

$$\text{Area} = \int_0^A \left(\frac{x}{3} - (-x^3)\right)\mathrm{d}x + \int_A^B \left(\frac{x}{3} - \ln x\right)\mathrm{d}x$$
$$\approx 0.393\ \text{units}^2$$

Integrating With Respect to y

Sometimes it is simpler to regard a region as being bounded by functions of y instead of x.

Theorem 5.8. If a region is described by the inequalities $c \le y \le d$ and $g(y) \le x \le f(y)$, then its net area is

$$\int_c^d (f(y) - g(y))\,\mathrm{d}y$$

The common advice for these is 'right minus left' as opposed to 'top minus bottom'. Shift your perspective for these types of problems! The primary difficulty in the setup of these problems is understanding that the limits of the integral are *not* x bounds, they are y bounds! So when finding the intersections of curves to get the limits of the integral, make sure you equate the functions of y.

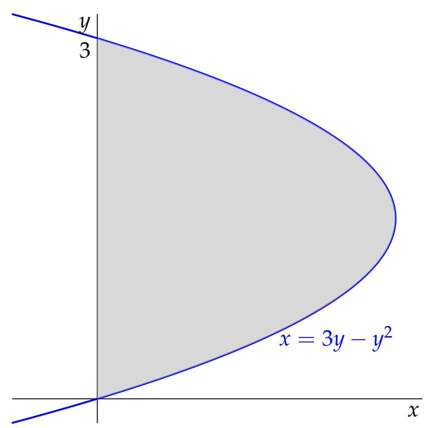
This approach is advantageous when the bounds of a region are given by curves which are impossible to write as a function of x, or otherwise would create a very ugly integral expression in terms of x.

Examples 5.9. 1. Find the area of the region bounded by the curve $x = 3y - y^2$ and the y-axis.

Let's first sketch the region in question (shown on the right). We decide to integrate with respect to y in this question. To find the y bounds, we need to find the y-intercepts:

$$3y - y^2 = y(3-y) = 0 \implies y = 0, 3$$

In this situation, we may imagine that the 'right' graph is the parabola, and the 'left' being the y-axis, i.e. $x = 0$. So



$$\begin{aligned}\text{Area} &= \int_0^3 (3y - y^2)\,\mathrm{d}y = \frac{3}{2}y^2 - \frac{1}{3}y^3\Big|_0^3 \\ &= \frac{3}{2}(3)^2 - \frac{1}{3}(3)^3 = \frac{27}{2} - 9 = \frac{9}{2}\text{ units}^2\end{aligned}$$

Compare this to the alternative method of isolating y:

$$\begin{aligned}&y^2 - 3y = -x \implies y^2 - 3y + \frac{9}{4} = \left(y - \frac{3}{2}\right)^2 = -x + \frac{9}{4} \\ &\implies y = \frac{3}{2} \pm \sqrt{-x + \frac{9}{4}}\end{aligned}$$

where the positive square root is the upper half of the parabola, and the negative is the lower half, which intersect at $x = \frac{9}{4}$. The area is thus

$$\text{Area} = \int_0^{9/4} \left[\frac{3}{2} + \sqrt{-x + \frac{9}{4}} - \left(\frac{3}{2} - \sqrt{-x + \frac{9}{4}}\right)\right] \mathrm{d}x$$

but who would ever want to evaluate this?!

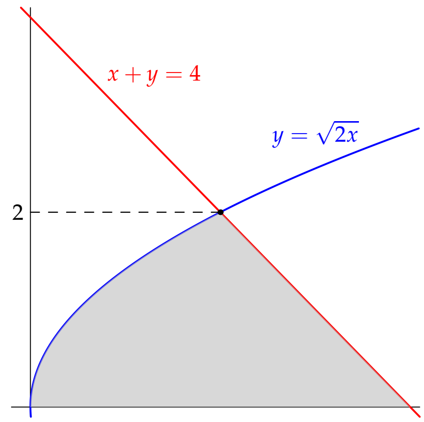
2. Find the area bounded by the curves $y = \sqrt{2x}$, $x + y = 4$, and the x-axis.

   We decide to integrate with respect to y. The equations in terms of y are $x = \frac{1}{2}y^2$ and $x = 4 - y$. The y-coordinate of the intersection is

   $$\frac{1}{2}y^2 = 4 - y \Longrightarrow y^2 + 2y - 8 = (y+4)(y-2) = 0$$

   The intersections occur when $y = -4$ and $y = 2$. The point where $y = -4$ is below the x-axis, so we ignore it. Thus

   $$\begin{aligned} \text{Area} &= \int_0^2 (4 - y - \frac{1}{2}y^2)\,\mathrm{d}y = 4y - \frac{1}{2}y^2 - \frac{1}{6}y^3 \Big|_0^2 \\ &= 4(2) - \frac{1}{2}(2)^2 - \frac{1}{6}(2)^3 - 0 = \frac{14}{3}\ \text{units}^2 \end{aligned}$$

   We could have also computed the area in terms of x, but we would have had to use two integrals.

Average Value

Recall the usual meaning of *average*: if we have a collection of n values $y_1, \ldots, y_n$, then its average is

$$y_{\text{av}} = \frac{y_1 + \cdots + y_n}{n}$$

Now observe that a Riemann sum for a function f on an interval $[a, b]$ is simply the average value of the rectangle-heights $f(x_i^*)$, multiplied by the length of $[a, b]$:

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = (b-a)\sum_{i=1}^{n} \frac{f(x_i^*)}{n}$$

This motivates us to *define* the notion of average for any integrable function.

Definition 5.10. The average value of a function f over $[a, b]$ is

$$f_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

To visualize the average value, remember that $\int_a^b f(x)\,\mathrm{d}x$ is the signed area between the graph of f and the x-axis over $[a, b]$. The region has a 'base' of $(b-a)$, so if we divide the area of the region by $(b-a)$, we get the average height of the region, i.e. the average value of f.

Note that if f is a constant function $f(x) = c$, then the average value is simply

$$f_{\text{av}} = \frac{1}{b-a}\int_a^b c\,\mathrm{d}x = \frac{1}{b-a}(b-a)c = c$$

Examples 5.11. 1. If $f(x) = x^2 - 3x + 2$, find the average value of f on $[0, 4]$.

$$f_{\text{av}} = \frac{1}{4-0}\int_0^4 (x^2 - 3x + 2)\,\mathrm{d}x = \frac{1}{4}\left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\Big|_0^4 = \frac{4}{3}$$

2. Find the average value of $g(x) = \sin 2x$ over $[\frac{\pi}{6}, \frac{\pi}{2}]$.

$$g_{\text{av}} = \frac{1}{\frac{\pi}{2} - \frac{\pi}{6}}\int_{\pi/6}^{\pi/2} \sin 2x\,\mathrm{d}x = \frac{3}{\pi}\left(-\frac{1}{2}\cos 2x\right)\Big|_{\pi/6}^{\pi/2} = \frac{9}{4\pi}$$

The definition of average value allows us to re-imagine some of our previous concepts. For instance, the *average rate of change* of f over $[a, b]$ would be the average value of the rate function of f, i.e. the average value of the *derivative* f':

$$f'_{\text{av}} = \frac{1}{b-a}\int_a^b f'(x)\,\mathrm{d}x = \frac{f(b) - f(a)}{b-a}$$

which is exactly our previous definition. Furthermore, we could play the same game with the *average value of velocity* over time interval $t = a$ to $t = b$:

$$v_{\text{av}} = \frac{1}{b-a}\int_a^b v(t)\,\mathrm{d}t = \frac{s(b) - s(a)}{b-a}$$

since the position function s is the anti-derivative of v. Again, this is identical to our previous definition of average velocity.

The Mean Value Theorem for Integrals

It seems obvious that if a function is continuous on an interval, then its value must at some point equal its average. Consider the following real-world examples.

1. There will always be some time when your speed equals your average speed for a journey.
2. At some point during the day, the temperature will equal the average temperature over that day.

This is indeed a Theorem.

Theorem 5.12 (Mean Value Theorem for Integrals). If f is continuous on $[a, b]$, then there exists a number c between a and b such that

$$f(c) = f_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Example 5.11.1 cont. For $f(x) = x^2 - 3x + 2$ on $[0, 4]$, we have $f_{\text{av}} = \frac{4}{3}$. We can find the values c on that interval which satisfy MVT for integrals:

$$f(c) = \frac{4}{3} \implies c^2 - 3c + \frac{2}{3} = 0 \implies c = \frac{3 \pm \sqrt{19/3}}{2} \approx 2.758, 0.242$$

Exercises 5.1. 1. Find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

2. Let R be the region enclosed by the graph of $y = e^x$, the horizontal line $y = 5$, and the y-axis.
   (a) Find the area of R by integrating with respect to x.
   (b) Find the area of R by integrating with respect to y.
3. Find the area of the region bounded by $y = 4\sqrt{2x}$, $y = 2x^2$, and $y = -4x + 6$.
4. Find the area enclosed by $x = y^3 - 5y$ and $x = 2y^2 - 6$.
5. The area below the graph of $g(x) = \frac{1}{x}$ from $x = 1$ to $x = k$ is 3 units^2, for some constant k. Find the value of k.
6. (Calculator) Find the area of the region enclosed by the graphs of $f(x) = -\frac{1}{2}\left(x^3 + x^2 - 5x\right)$ and $g(x) = \frac{1}{2}x$.
7. Find the area of the region bounded by the curve $x = -\sqrt{13 - y^2}$ and the y-axis.
   (*Hint: No anti-derivative is required here!*)
8. Find the area of the triangle whose vertices in the xy-plane have coordinates $(4, 5)$, $(-1, 3)$, and $(2, 2)$. For this question, use integrals rather than geometry.
   (*Hint: Find the equations of the lines connecting the points.*)
9. Write, but do not evaluate, an integral that gives the area of the region bounded by the right-side of the hyperbola $x^2 - 2y^2 = 1$ and the vertical line $x = 3$.
10. Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.
11. Let h be the function defined by

$$h(x) = \frac{1}{\sqrt{1 - x^2}}$$

   (a) Show that the graph of h is symmetric about the y-axis. That is, h is an *even* function.
   (b) Show that h has domain $-1 < x < 1$.
   (c) Find the exact area enclosed by the graph of h and the x-axis from $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.
12. Find the average values of the following functions on the given intervals.
   (a) $f(x) = x^{-3}$ on $[2, 5]$
   (b) $g(x) = \cos x$ on $[0, \frac{3\pi}{2}]$
   (c) $h(x) = \begin{cases} x^2, & 1 \leq x \leq 3 \\ e^x, & 3 < x \leq 6 \end{cases}$ over $[1, 6]$
13. (Hard) Without evaluating integrals, explain why, for all positive integers n,

$$\int_0^1 x^n \, dx + \int_0^1 x^{1/n} \, dx = 1$$

5.2 Return to Motion and Other Contextual Applications

The primary usage of differentiation in real-life applications was to do with *rates of change*. For integrals, their primary usage is to determine *net change* of a function over some interval.

Armed with some knowledge of integration, we make a return to motion of an object. The oldest, and perhaps the biggest motivator of calculus is the distance traveled by an object whose velocity is known.

Recall the *position function* of an object P is $s(t)$. The first and second derivative of $s(t)$ are, respectively, the *velocity function* $v(t)$ and the *acceleration function* $a(t)$.

Definition 5.13. The *displacement* of object P from $t = a$ to $t = b$ is the distance between the object's position at $t = b$ and its position at $t = a$:

$$\text{displacement} = s(b) - s(a)$$

Example 5.14. A biker begins her ride 45 miles from her house at 1 p.m. At 3 p.m., she is 75 from her house. If we let $t = 0$ be noon and the function $s(t)$ the position of the biker relative to her house in miles, then then $s(1) = 45\,\text{mi}$ and $s(3) = 90\,\text{mi}$. Thus her *displacement* from 1 p.m. ($t = 1$) to 3 p.m. ($t = 3$) is

$$\text{displacement} = s(3) - s(1) = 75 - 45 = 30\,\text{mi}$$

However, we have the knowledge that the velocity function is the derivative of the position function, i.e. the position function is the *anti-derivative* of velocity. This along with FTC part II allows us to obtain the following result.

Theorem 5.15. The displacement of object P from $t = a$ to $t = b$ is the definite integral of the object's velocity function from $t = a$ to $t = b$:

$$\text{displacement} = \int_a^b v(t)\,\mathrm{d}t$$

Proof. By the Fundamental Theorem of Calculus part II, we have

$$\int_a^b v(t)\,\mathrm{d}t = s(b) - s(a) = \text{displacement}$$

since $s(t)$ is the anti-derivative of $v(t)$. ■

The Theorem is most useful when we are given either the graph of the velocity function or the equation for $v(t)$ itself, but not the equation or graph of $s(t)$.

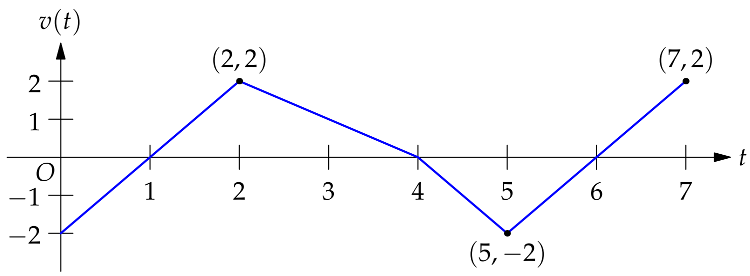
Take note of the *units* in the next example. Remember that differentiating a function divides by the unit of the variable we differentiate with respect to. When *integrating*, we *multiply* by the unit of the variable we integrate with respect to. For instance, if $s(t)$ is measured in feet and t in hours:

Differentiate: $v(t)\ \text{ft/h} \rightsquigarrow a(t)\ \text{ft/h}^2$ Integrate: $v(t)\ \text{ft/h} \rightsquigarrow s(t)\ \text{ft}$

Examples 5.16. 1. A particle moving along a horizontal axis has velocity $v(t) = 3t^2 - 2t + 1$ meters per second. Find the displacement of the particle from $t = 2$ to $t = 4$.

$$\text{displacement} = \int_2^4 (3t^2 - 2t + 1)\,\mathrm{d}t = t^3 - t^2 + t\Big|_2^4 = 46\,\mathrm{m}$$

2. A particle travels along a vertical axis. Provided below is the graph of $v(t)$, the velocity of the particle, measured in feet per second, t seconds after the object first moves. Find the displacement of the particle from $t = 3\,\mathrm{s}$ to $t = 6\,\mathrm{s}$.



Graph of v

The displacement of the object is

$$\text{displacement} = \int_3^6 v(t)\,\mathrm{d}t = \underbrace{\int_3^4 v(t)\,\mathrm{d}t}_{\text{area of upper }\triangle} + \underbrace{\int_4^6 v(t)\,\mathrm{d}t}_{\text{area of lower }\triangle} = \frac{1}{2} - 2 = -\frac{3}{2}\,\mathrm{ft}$$

We keep in mind that, for definite integrals without absolute value symbols, the area is *signed*, so regions below the horizontal axis are regarded to have negative area.

The interpretation of this result is, at $t = 6\,\mathrm{s}$, the particle is 1.5 ft lower than it was at $t = 3\,\mathrm{s}$.

The idea of displacement is in contrast to *total distance traveled*.

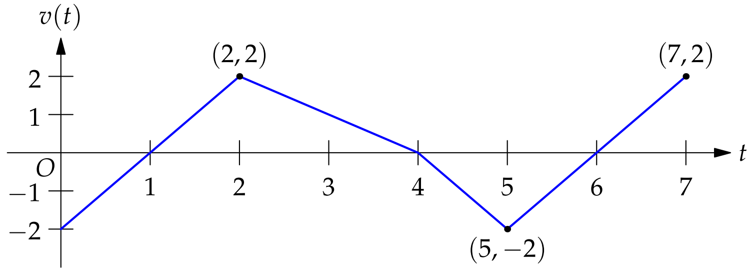
Definition 5.17. The *total distance traveled* by object P between $t = a$ and $t = b$ is the total area between the graph of velocity $v(t)$ and the t-axis. Otherwise said, it is the definite integral from $t = a$ to $t = b$ of the *speed*:

$$\text{Total distance} = \int_a^b |v(t)|\,\mathrm{d}t$$

Why is total distance traveled defined as such? Say, for example, that I walk two meters forward, then walk two meters backward. My *displacement* over that time interval is 0 m, since the net change in my position was 0; I am in the same position as when I started. But my *total distance traveled* was 4 m, because I still walked a total of 4 meters, regardless of my direction of travel.

Notice the equivalence of this idea to that of *speed*, which is defined as simply the numerical quantity of velocity, disregarding direction of travel.

Example 5.16.2 cont. We use the same graph of $v(t)$. This time, we find the total distance traveled by the particle from $t = 3\,\text{s}$ to $t = 6\,\text{s}$.



Graph of v

$$\text{Total distance} = \int_3^6 |v(t)|\,\mathrm{d}t = \int_3^4 v(t)\,\mathrm{d}t - \int_4^6 v(t)\,\mathrm{d}t = \frac{1}{2} + 2 = \frac{5}{2}\,\text{ft}$$

Suppose we want to find, rather than the change in position after a period of time, the *exact* position of an object after some time period given its velocity. This is only possible if we are given an *initial condition*, which is information about the position of the object at some point in time.

Theorem 5.18. The position $s(b)$ of an object at $t = b$ given $s(a)$ is

$$s(b) = s(a) + \int_a^b v(t)\,\mathrm{d}t$$

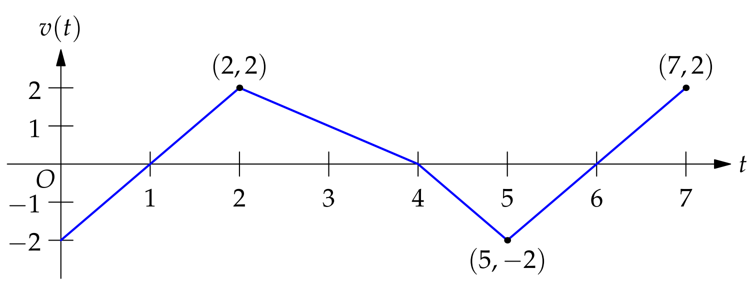
Proof. This is a simple application of the Fundamental Theorem:

$$\int_a^b v(t)\,\mathrm{d}t = s(b) - s(a) \implies s(b) = s(a) + \int_a^b v(t)\,\mathrm{d}t$$

as desired. ∎

Crucially, this method is not limited to velocity and position: the value of *any* anti-derivative F can be found so long as we are given an initial condition (the value of the anti-derivative at some point $F(a)$) and the graph or equation of the derivative of said function f.

Examples 5.19. 1. Example 5.16.2 cont. Suppose the particle has position $s(4) = 5\,\text{m}$. Again, the graph of velocity $v(t)$ is provided below. Find $s(0)$ and $s(7)$.



Graph of v

To find $s(0)$, we use the initial condition $s(4) = 5$, Since $0 < 4$, we must switch the bounds of the integral and introduce a minus sign.

$$s(0) = s(4) + \int_4^0 v(t)\,\mathrm{d}t = s(4) - \int_0^4 v(t)\,\mathrm{d}t = 5 - \left(\int_0^1 v(t)\,\mathrm{d}t + \int_1^4 v(t)\,\mathrm{d}t\right)$$
$$= 5 - (-1 + 4) = 2\,\mathrm{m}$$

where $\int_0^1 v(t)\,\mathrm{d}t$ is the negative area of the triangle created by the graph from $t = 0$ to $t = 1$, and $\int_1^4 v(t)\,\mathrm{d}t$ is the area of the triangle from 1 to 4. To find $s(7)$:

$$s(7) = s(4) + \int_4^7 v(t)\,\mathrm{d}t = 5 + \int_4^6 v(t)\,\mathrm{d}t + \int_6^7 v(t)\,\mathrm{d}t = 5 - 2 + 1 = 4\,\mathrm{m}$$

2. Suppose $f'(x) = 2x^2 + 4x - 1$ and $f(0) = 3$. Find $f(2)$.

   There are two methods. We could use the formula described in Theorem 5.18:

$$f(2) = f(0) + \int_0^2 f'(x)\,\mathrm{d}x = 3 + \left[\frac{2}{3}x^3 + 2x^2 - x\right]_0^2 = 3 + \frac{34}{3} = \frac{43}{3}$$

   Or, we could find the formula for $f(x)$ by finding the indefinite integral of $f'(x)$:

$$f(x) = \int f'(x)\,\mathrm{d}x = \int (2x^2 + 4x - 1)\,\mathrm{d}x = \frac{2}{3}x^3 + 2x^2 - x + C$$

   Then we can use the initial condition $f(0) = 3$:

$$f(0) = 3 \Longrightarrow \frac{2}{3}(0)^3 + 2(0)^2 - 0 + C = 3 \Longrightarrow C = 3 \Longrightarrow f(x) = \frac{2}{3}x^3 + 2x^2 - x + 3$$

   Then we can find $f(2)$:

$$f(2) = \frac{2}{3}(2)^3 + 2(2)^2 - 2 + 3 = \frac{43}{3}$$

The second method we used in the previous example is the same we will use for solving differential equations later. It is called an *initial value problem*.

Other Real-Life Applications

The techniques discussed in this section can be used to solve many real-life problems. Importantly, it is guaranteed that one free-response question in the calculator section will be of this variety! A few notes:

1. The problem will introduce a function. This function could either represent a *quantity* or *rate*. Read the question carefully to determine which it is! The units will also always be provided, which may also be a giveaway.

2. Remember the relationships between functions. Differentiate to obtain a rate, and integrate to obtain a quantity.

3. Units are everything! Provide correct units during explanations, and use them to decide whether to integrate, differentiate, or simply evaluate the given function.

4. Directly quoting the relevant Definitions and Theorems is sufficient explanation. No need to write an essay!

This type of problem is the pinnacle of this course. All manners of questions may be asked, ranging from differentiation to integration. Be polished up on all required knowledge, and do many examples to master!

Examples 5.20. 1. (Calculator) On a certain day, customers enter a 24-hour clothing shop at a rate modeled by the function E given by

$$E(t) = 15 - 15\cos\left(\frac{\pi t}{12}\right)$$

Customers leave the shop at a rate modeled by the function L given by

$$L(t) = 3 + \ln\left(0.2t^2 + 1\right)$$

Both $E(t)$ and $L(t)$ are measured in people per hour, and t is measured in hours since 8 a.m. ($t = 0$). At 8 a.m., there are 20 customers in the store.

(a) How many people enter the shop over the 6-hour period from 8 a.m. ($t = 0$) to 2 p.m. ($t = 6$)? Give your answer to the nearest whole number.

(b) What is the average number of customers that leave the shop over the 6-hour period from 8 a.m. ($t = 0$) to 2 p.m. ($t = 6$)?

(c) To the nearest whole number, what is the greatest number of customers in the store on the time interval $0 \leq t \leq 24$? Justify your answer.

(d) Is the rate of change in the number of people in the store increasing or decreasing at 2 p.m. ($t = 6$)? Explain your reasoning

Sample response:

(a) Since E models the rate at which people enter the store, we calculate

$$\int_0^6 E(t)\,\mathrm{d}t = 32.704220$$

To the nearest whole number, 33 people enter the lake from 8 a.m. to 2 p.m.

(b) Since L represents the rate at which people leave the store, we calculate the average value

$$L_{\text{av}} = \frac{1}{6-0}\int_0^6 L(t)\,\mathrm{d}t = 4.009043$$

The average number of customers that leave the shop per hour from 8 a.m. to 2 p.m. is 4.009 customers per hour.

(c) We are looking for the *absolute maximum* number of customers in the shop on $[0, 24]$. To do so, we let the function A represent the number of customers in the store at time t hours:

$$A(t) = 20 + \int_0^t (E(s) - L(s))\,\mathrm{d}s$$

We need to first find the critical values of function A:

$$A'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(20 + \int_0^t (E(s) - L(s))\,\mathrm{d}s\right) = E(t) - L(t) = 0$$
$$\implies t = B = 2.8408999, \quad t = C = 20.030229$$

t	$A(t)$
0	20
B	14.207918
C	255.205329
24	235.263144

Therefore the greatest number of customers in the store over the 24-hour period is approximately 255 people.

(d) $A''(t) = E'(t) - L'(t) \implies A''(6) = E'(6) - L'(6) = 3.6343 > 0$

Because $E'(6) - L'(6) > 0$, the rate of change in the number of customers is increasing at time $t = 6$ hours.

A few notes:

- The question specifies that functions E and L are *rates*. Another tell is that both functions have the unit people per hour.
- In part (a), the question 'how many people enter the shop...' can be rephrased as 'what is the *net change* in the number of people entering from $t = 0$ to $t = 6$', hence why we integrate. Another giveaway is that we are asked for the number of people, whereas $E(t)$ is measured in people per hour. To remove the 'per hour', we integrate to multiply by the unit 'hours'.
- Anytime we are asked for *average*, our minds should go to the average value formula. So we find the average value of L over $[0, 6]$. We give our answer in the unit customers per hour because, even though the integral removes the 'per hour', division by $(6 - 0)$ re-introduces it (since the unit for $6 - 0$ is hours).
- The words 'maximum' or 'minimum' along with an interval should signal absolute extrema. Since we need the maximum number of people in the store, we need a function which models the number of customers in the store. For the same reason mentioned previously, we need to integrate $E - L$ to get the unit to 'customers'. Because at $t = 0$ there are 20 people in the store, we add 20 to our accumulation function. On account of FTC part I, differentiating the integral expression in $A(t)$ removes it, and we simply need to find the zeros of $E(t) - L(t)$. The table of values is acceptable justification for this problem, so long as we remember to consider the endpoints of $[0, 24]$ as well.
- In part (d), we are asked whether the *rate* is increasing or decreasing, which means we need the rate *of the rate of change* of customers in the store at $t = 6$, which is $A''(6)$. Using the function A in our explanation is not strictly necessary, so long as we consider $E'(6)$ and $L'(6)$; another sufficient justification would be that $E'(5) > L'(5)$, which is equivalent to the explanation provided in the sample.

2. (Calculator) The velocity of a particle, P, moving along the x-axis is modeled by the twice-differentiable function v, given by

$$v(t) = \sqrt{t+5} \cdot \sin^3(0.34t)$$

where $v(t)$ is measured in meters per second and t is measured in seconds. At time $t = 0$, particle P is at position $x(0) = 3$.

(a) Find the values of $v(4)$ and $v(12)$. Use these to explain why, on the interval $4 < t < 12$, there must be a time for which particle P is at rest.

(b) At time $t = 6$, is particle P speeding up or slowing down? Justify your answer.

(c) Find the total distance traveled by particle P over the interval $0 \leq t \leq 12$.

(d) What is the position of the particle at $t = 12$ seconds?

Sample response:

(a) $v(4) \approx 2.805$ and $v(12) \approx -2.164$. Since v is twice-differentiable $\Longrightarrow$ v is continuous and $v(4) > 0$ and $v(12) < 0$, by the Intermediate Value Theorem, there must be a time c on $(4, 12)$ for which $v(c) = 0$.

(b) $v(6) \cdot v'(6) \approx -2.612 < 0$, so P is slowing down at $t = 6$.

(c) First, $v(t) = 0 \Longrightarrow t = A = 9.239978$, whence $v(t) > 0$ on $[0, A]$ and $v(t) < 0$ on $[A, 12]$.

$$\text{Total distance} = \int_a^b |v(t)| \, \mathrm{d}t = \int_0^A v(t) \, \mathrm{d}t - \int_A^{12} v(t) \, \mathrm{d}t \approx 13.848 \, \mathrm{m}$$

(d) We are finding $x(12)$:

$$x(12) = x(0) + \int_0^{12} v(t) \, \mathrm{d}t = 4 + \int_0^{12} v(t) \, \mathrm{d}t \approx 14.408 \, \mathrm{m}$$

A few notes:

- We should know by now that $v(t)$ is the rate of change of an object's position at time t.
- Refer to the definition of 'at rest': this means the velocity of P is zero. Using IVT is appropriate here.
- Individually calculating $v(6)$ and $a(6) = v'(6)$ then stating that they have different signs is also a sufficient justification for P slowing down.
- The reason we are finding the zeros of $v(t)$ on $[0, 12]$ is because total distance traveled by an object is the integral of the *speed*, which is equivalent to finding total area. Thus, we need to subtract the areas for which $v(t) < 0$.
- To $x(12)$, we use $x(0)$ plus the displacement of P over the time interval $0 \leq t \leq 12$. Vitally, the displacement is not the value of the integral we found in part (c)!

3. (Calculator) The quantity of a certain toxic chemical in a vat of liquid is modeled by a strictly decreasing, twice-differentiable function K, where $K(t)$ is measured in grams and t is measured in minutes. The chemical is slowly filtered out of the vat over 45 minutes, beginning at time $t = 0$. Values of $K(t)$ at selected times t for the first 30 minutes are given in the table below.

t (minutes)	0	10	18	25	30
$K(t)$ (grams)	32.0	26.1	23.3	19.8	15.0

(a) Use the data in the table to estimate $K'(14)$. Show the working that leads to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{30} K'(t)\,dt$. Using correct units, interpret the meaning of this value in the context of this problem.

(c) For $0 \le t \le 30$, the average quantity of chemical in the vat is $\frac{1}{30}\int_0^{30} K(t)\,dt$. Use a right Riemann sum with the four sub-intervals indicated by the data in the table to approximate $\frac{1}{30}\int_0^{30} K(t)\,dt$. Does this approximation overestimate or underestimate the average amount of chemical in the container over these 30 minutes? Explain your reasoning.

(d) For $30 \le t \le 45$, the function K which models the quantity of chemical has first derivative given by $K'(t) = -1.03 + 18e^{-0.09t}$. Based on this model, what amount of chemical remains in the vat at $t = 45$ minutes?

Sample response:

(a) We find the average rate of change of K over $[10, 18]$ to estimate $K'(14)$:

$$K'(14) \approx \frac{K(18) - K(10)}{18 - 10} = \frac{23.3 - 26.1}{8} = -0.35$$

The amount of chemical in the vat at time $t = 14$ minutes is decreasing at a rate of approximately 0.35 grams per minute.

(b) Using FTC part II:

$$\int_0^{30} K'(t)\,dt = K(30) - K(0) = -17.0$$

17.0 grams of toxic chemical have been filtered out of the vat from $t = 0$ to $t = 30$ minutes.

(c) Using a right Riemann sum:

$$\begin{aligned}\frac{1}{30}\int_0^{30} K(t)\,dt &\approx \frac{1}{20}(10 \cdot K(10) + 8 \cdot K(18) + 7 \cdot K(25) + 5 \cdot K(30)) \\ &= \frac{1}{20}(10 \cdot 26.1 + 8 \cdot 23.3 + 7 \cdot 19.8 + 5 \cdot 15.0) = \frac{1}{20} \cdot 661.0 = 33.05\end{aligned}$$

This approximation is an underestimate because a right sum is used and the function K is strictly decreasing.

(d) We are finding $K(45)$:

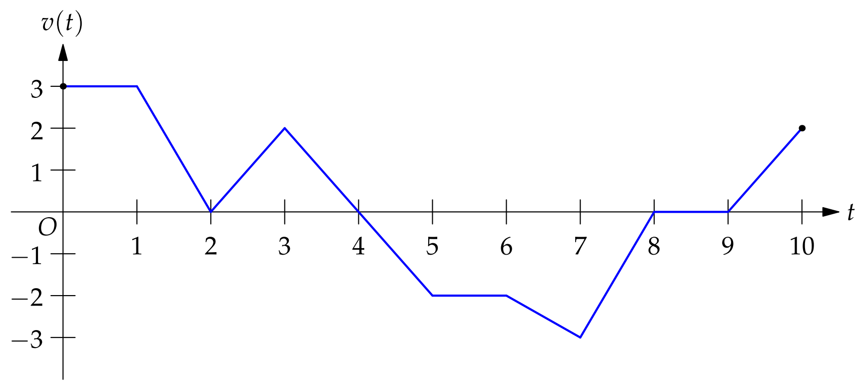
$$K(45) = K(30) + \int_{30}^{45} K'(t)\,dt = 15.0 + \int_{30}^{45} (-1.03 + 18e^{-0.09t})\,dt = 9.507$$

A few notes:

- The function K is a *quantity* rather than a rate, as indicated by the units.
- We have encountered a problem similar to the one in part (a) before: we use average rate of change to estimate the value of the instantaneous rate of change of K at $t = 14$. Since K is measured in grams and t in minutes, the unit for $K'(t)$ should be the unit of K divided by the unit of t, hence grams per minute.
- We plainly recognize the application of FTC part II here. Recall that the definite integral of a function over an interval represents the net change of the value of an anti-derivative of the function over that interval. Here, K is the anti-derivative of K', so the net change of K over $[0, 30]$ in the context of this problem is the change in quantity of chemical.
- Don't forget to divide by 20 after finding the Riemann sum! We aren't asked to interpret our answer here. Refer to a previous Theorem to justify that the right Riemann sum of a decreasing function is an underestimate. It may be worth it to draw a picture to convince yourself of this if you forget the exact Theorem.
- We use the initial value $K(30) = 15.0$ given in the table for the calculation. We integrate the *rate* K' to achieve the *quantity* K at some time.

These are just several examples, and there are more in the exercises below, but remember that the free-response questions of every AP exam from previous years are provided for free on the College Board website. So if you need more practice, you can always search online.

Exercises 5.2. 1. A particle travels along a horizontal axis. Its velocity at time t seconds is given by the function $v(t)$, measured in feet per second. At $t = 0$, the particle is at position $s(0) = 1$ ft. A graph of $v(t)$ on the interval $0 \le t \le 10$ is shown below. For each part, provide correct units in your answer.



Graph of v

(a) For each $t = 2$, $t = 4$, and $t = 10$, find $s(t)$, $v(t)$, and $a(t)$, if such a value exists.

(b) On what intervals for $0 \le t \le 10$ is the particle moving to the right? Left? At rest?

(c) On what intervals for $0 \le t \le 10$ is the particle speeding up? Slowing down?

(d) At what times for $0 \le t \le 10$ does the particle change direction?

(e) At what times for $0 \le t \le 10$ does the graph of $s(t)$ have a point of inflection?

(f) Over $0 \le t \le 10$, where is the furthest right the particle gets? And when does this occur? Repeat for furthest left.

(g) Sketch a possible graph of $s(t)$ and $a(t)$ for $0 \le t \le 10$.

(h) Find the displacement of the particle over $[0, 10]$.

(i) Find the total distance traveled by the particle over $[0, 10]$.

(j) Find the average velocity of the particle over $[0, 10]$.

(k) Find the average acceleration of the particle over $[0, 10]$.

2. (Calculator) A fungal culture is inserted into a laboratory dish, where it grows filaments which spread throughout the dish. For $0 \le t \le 14$, the total area covered by the mycelium is given by

$$A(t) = 0.933(1.22)^t$$

where $A(t)$ is measured in square centimeters and t is measured in days.

(a) Find the value of $A'(8)$. Using correct units, interpret the meaning of the value in the context of the problem.

(b) Find the time t for which the area covered by the mycelium in the dish is equal to the average area covered by the mycelium in the dish over the interval $0 \le t \le 14$.

(c) After two weeks in the laboratory dish ($t = 14$), the filaments stop growing and begin decaying at a rate of $D(t) = 6 - \ln(0.56t^3)$ square centimeters per day. Find the area covered by the mycelium in the dish after three weeks ($t = 21$).

(d) Explain why, for $0 \le t \le 14$, the linear approximation to A at $t = 8$ would underestimate the the area covered by the mycelium in the dish.

3. (Calculator) On a certain day, the rate at which petroleum is pumped into an oil drum is modeled by the function R, given by

$$R(t) = 408\sin(7.22 - 0.103t), \quad 0 \le t \le 9$$

where $R(t)$ is measured in liters per hour and t is measured in hours elapsed since midnight. Over $0 \le t \le 9$, petroleum is also extracted from the oil drum at a constant rate of 130 liters per hour. At midnight ($t = 0$), the oil drum has 2000 liters of oil in it.

   (a) Find $R'(4)$. Using correct units, interpret this value in the context of the problem.

   (b) Find the total volume of petroleum that is pumped into the drum from midnight to 9 a.m.

   (c) Is the volume of oil in the drum increasing or decreasing at time $t = 6$ hours? Show the work leads to your answer.

   (d) A label on the oil drum warns that it is at risk of overflowing when the volume reaches 2500 liters. Is there any time between midnight and 9 a.m. for which the oil drum is at risk of overflowing?

4. (Calculator) A certain species of tree grows according to the strictly decreasing, differentiable function G, where $G(t)$ is measured in feet per year and t is measured in years after the sapling is planted. When the sapling is planted, it is 3 feet tall. Selected values of $G(t)$ are given in the following table.

t (years)	0	1	3	4	7
$G(t)$ (feet per year)	14.2	8.3	6.1	5.3	4.5

   (a) Use a left Riemann sum with the four sub-intervals indicated in the table to approximate the value of $\int_0^7 G(t)\,dt$. Using correct units, interpret this value in context of the problem.

   (b) Is the approximation in part (a) greater than or less than $\int_0^7 G(t)\,dt$? Justify your answer.

   (c) Explain why there must exist a value c, for $1 < c < 4$, such that $G'(c) = -1$ feet per year per year.

   (d) The growth of the tree may be amplified with the use of fertilizer. With fertilizer, the rate at which the tree's height changes can be modeled by $F(t) = 0.2t^2 - 2.8t + 14.7$ for $0 \le t \le 7$. By using your answer in part (a), after 7 years, approximately how much taller is the tree grown with fertilizer compared to the tree grown without fertilizer?

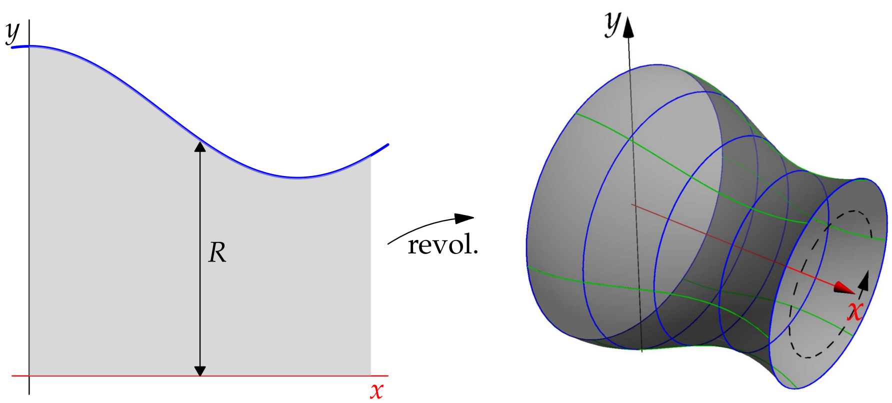
5.3 Volumes

For this section, we should remind ourselves of exactly what a definite integral is: a Riemann sum of infinitely many thin rectangles to find the *area* of a region. Naturally, we may be curious whether this process can be used to find *volumes*. The answer is yes![27]

Volumes of Solids of Revolution

Definition 5.21. A *solid of revolution* is generated by revolving a 2-dimensional region around a straight line, called an *axis of revolution*.

If we take a look at a picture of any solid of revolution, it becomes obvious that if we slice the solid with a plane perpendicular to the axis of revolution, the shape is a *circle*.



The region bounded by the graph of $f(x) = 2 + \frac{1}{2}\cos x$, the x-axis, and the vertical lines $x = 0$ and $x = 4$ revolved around the x-axis.

What does this mean for us? We can find the exact volume of the solid by finding a Riemann sum of the *volumes* of many thin *cylinders*, whose radius is the distance between the boundary of the region and the axis of revolution!

Theorem 5.22 (Disc Method). If a 2-dimensional bounded region over $a \leq x \leq b$ is revolved around a horizontal axis of revolution, then the volume of the resulting solid is given by

$$V = \int_a^b \pi R^2 \, dx = \pi \int_a^b R^2 \, dx$$

where R is the vertical distance between the boundary of the region and the axis of revolution.

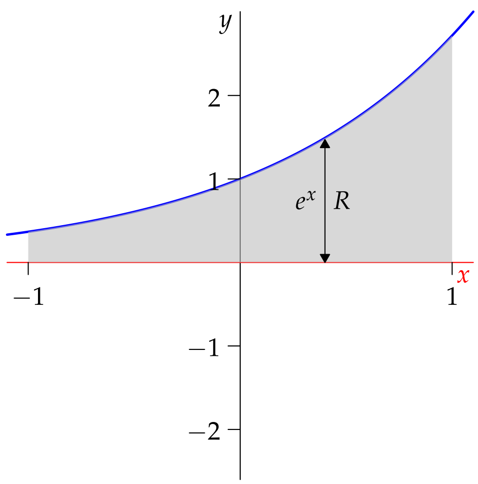
Recall that the volume of a cylinder is $\pi r^2 h$: here, we are treating $r = R$ as the distance between the axis of revolution and the boundary of the enclosed region in order to produce a Riemann sum of cylinders, whose heights are infinitesimally thin.
This approach is called the disc method because the cross sections of the solid perpendicular to the axis of revolution are discs (filled in circles).

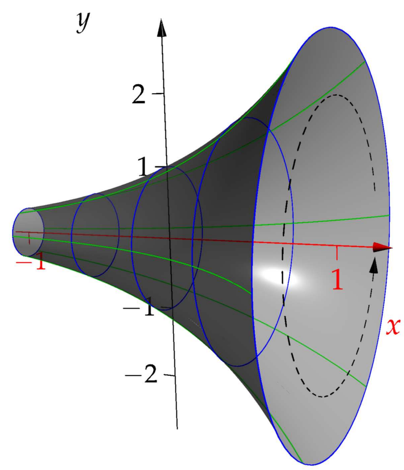
[27] Albeit with a catch: our current approach only works for solids whose cross sections are predictable. For volumes of more eccentric solids, we need *multivariable calculus* methods.

Examples 5.23. 1. Find the volume of the solid generated when the region enclosed by the curve $y = e^x$, the x-axis, and the lines $x = -1$ and $x = 1$ is revolved around the x-axis.

The x bounds are plainly -1 and 1. The distance between the x-axis and the boundary of the region is $R = e^x - 0 = e^x$. Therefore the volume of the solid is

$$V = \pi \int_a^b R^2 \, dx = \pi \int_{-1}^{1} e^{2x} \, dx = \pi \left[\frac{1}{2} e^{2x} \right]_{-1}^{1} = \frac{\pi}{2}(e^2 - e^{-2}) \text{ units}^3$$



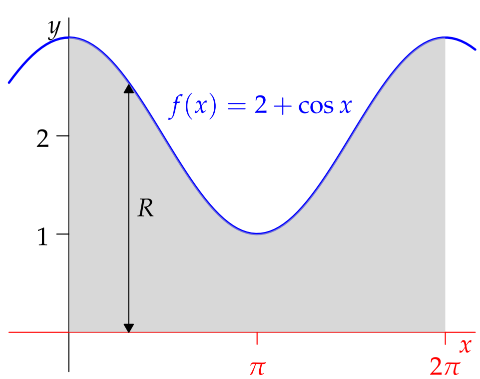


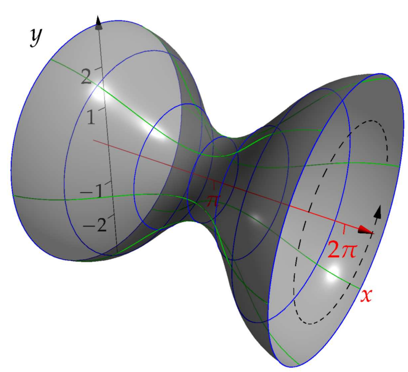
2. (Calculator) Find the volume enclosed when the graph of $f(x) = 2 + \cos x$ is rotated around the x-axis from $x = 0$ to $x = 2\pi$.

The x bounds are obviously $x = 0$ and $x = 2\pi$, and $R = f(x) = 2 + \cos x$. The volume of the solid is therefore

$$V = \pi \int_a^b R^2 \, dx = \pi \int_0^{2\pi} (2 + \cos x)^2 \, dx \approx 88.826 \text{ units}^3$$

The integral can be evaluated without a calculator (try it!).





Revolving About the y-axis

Sometimes we may rotate a bounded region about the y-axis. We still use the disc method, but the volume formula changes slightly.

Theorem 5.24. If a 2-dimensional bounded region over $c \le y \le d$ is revolved around a vertical axis of revolution, then the volume of the resulting solid is given by

$$V = \int_c^d \pi R^2 \, dy = \pi \int_c^d R^2 \, dy$$

where R is the horizontal between the boundary of the region and the axis of revolution.

In this scenario, we are integrating with respect to y, so, like with areas, we must have the region boundaries as functions of y. Also, c and d are *vertical* bounds, so make sure you do not make the mistake of using the incorrect limits.

Examples 5.25. 1. Find the volume of the solid generated when the area enclosed by the curves $x = \sqrt{y+1}$, $y = 3$, and the y-axis are revolved about the y-axis.

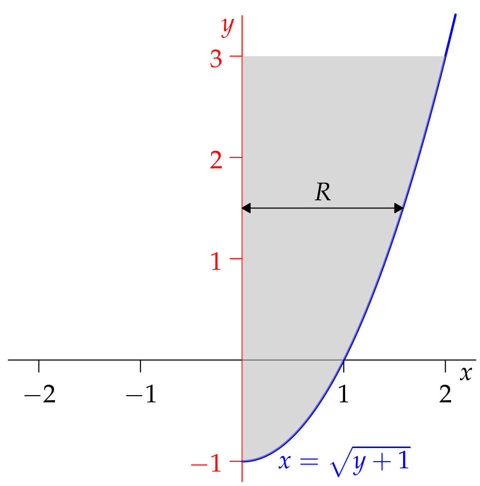
As we had with areas, we must find the bounds if not given. Here we are finding y bounds. The upper bound is given in the question by $y = 3$, and the lower bound is the y-intercept of the parabola:

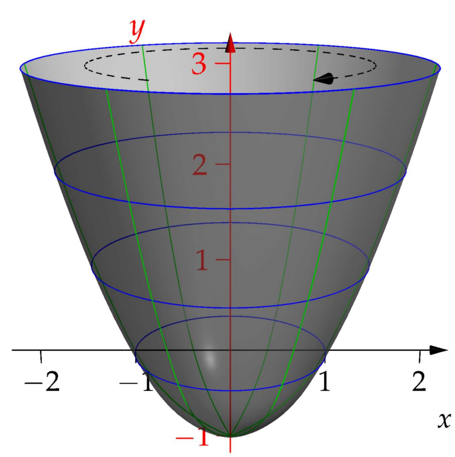
$$\sqrt{y+1} = 0 \implies y = -1$$

Thus the volume of the described solid is

$$V = \pi \int_{-1}^{3} \left(\sqrt{y+1}\right)^2 dy = \pi \int_{-1}^{3} (y+1)\, dy = \pi \left[\frac{1}{2}y^2 - y\right]_{-1}^{3} = 8 \text{ units}^3$$

If you are ever unsure about the bounds, drawing a picture can always help.





2. Find the volume of the solid generated when the area enclosed by the circle $x^2 + (y-2)^2 = 4$ in the first quadrant is revolved about the y-axis.

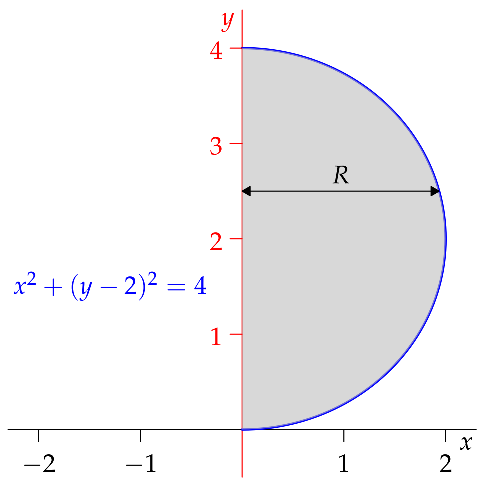
   Since the region is revolved around the y-axis, we need the vertical bounds for our integral. We should notice they come from the intersection between the circle and the y-axis. First, isolate x:

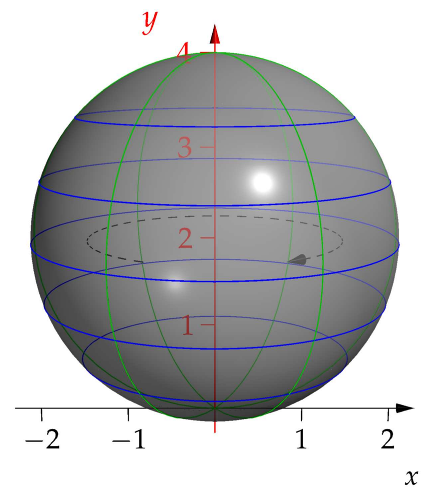
$$x^2 + (y-2)^2 = 4 \implies x^2 = 4y - y^2 \implies x = \sqrt{4y - y^2} \qquad \text{(find } x \text{ as function of } y\text{)}$$
$$\sqrt{4y - y^2} = 0 \implies 4y - y^2 = 0 \implies y = 0, 4 \qquad \text{(find } y \text{ bounds)}$$

   We ignore the negative square root since we work in the first quadrant. Then the volume is

$$V = \pi \int_0^4 \left(\sqrt{4y - y^2}\right)^2 \, dy = \pi \int_0^4 (4y - y^2) \, dy = \pi \left[2y^2 - \frac{1}{3}y^3\right]_0^4 = \frac{32\pi}{3} \text{ units}^3$$

   If you're clever, you may notice the resulting solid is simply a *sphere* of radius 2, whence the area is $V = \frac{4}{3}\pi(2)^3 = \frac{32\pi}{3}$ cubic units!





By now, we may now be recognizing that our method for computing volumes of solids of revolution is very similar to that of finding area: we get the bounds of a region by finding the appropriate intersections between curves. The major difference here is just the formula used.

The Washer Method

Recall the method of finding area of a region bounded by two curves several sections ago. There, we adhered to the advice 'top minus bottom' and integrated accordingly. A similar thing can be performed with solids of revolution, where the region being rotated is *not connected* to the axis of revolution.

Theorem 5.26 (Washer Method). If a 2-dimensional bounded region over $a \leq x \leq b$ is revolved around a horizontal axis of revolution, where the region is *not* connected to the axis of revolution, then the volume of the resulting solid is given by

$$V = \pi \int_a^b (R^2 - r^2)\,\mathrm{d}x$$

where R is the vertical distance between the further boundary of the region and the axis of revolution, and r is the vertical distance between the closer boundary of the region and the axis of revolution.
Moreover, we have the volume for such a solid revolved around a vertical axis:

$$V = \pi \int_c^d (R^2 - r^2)\,\mathrm{d}y$$

where R and r in this context are similarly defined.

This is not fully analogous to the 'top minus bottom' method we previously studied; the 'outer' and 'inner' distances are not necessarily the 'top' and 'bottom' curves respectively (or 'right' and 'left' if integrating with respect to y).
We call this the washer method because each cross section perpendicular to the axis of revolution are washers (circles with a hole).

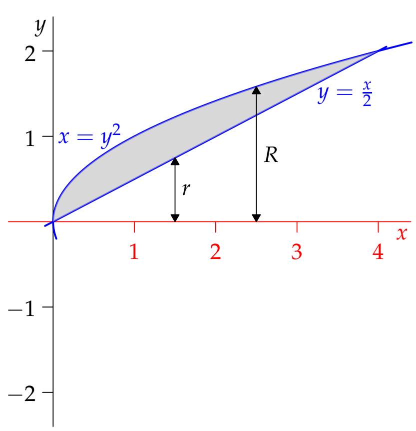
Examples 5.27. 1. Find the volume of the resulting solid when the region bounded by the curves $y = \frac{x}{2}$ and $x = y^2$ is revolved around the x-axis.

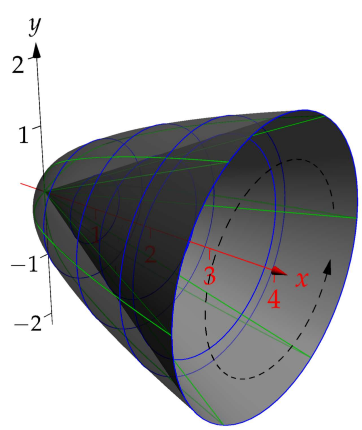
Since we are revolving around the x-axis, we integrate with respect to x, hence we need the equations of our curves as functions of x. The first is given, the second is $x = y^2 \implies y = \sqrt{x}$. We ignore the negative square root. We find the x-values of intersection:

$$\frac{x}{2} = \sqrt{x} \implies \frac{x^2}{4} = x \implies x^2 - 4x = 0 \implies x = 0, 4$$

The curve further away from the x-axis is the parabola, and the closer curve is the line. Thus the distances $R = \sqrt{x}$ and $r = \frac{x}{2}$. The volume of the solid is therefore

$$V = \pi \int_0^4 \left(\left(\sqrt{x}\right)^2 - \left(\frac{x}{2}\right)^2 \right) \mathrm{d}x = \pi \int_0^4 \left(x - \frac{1}{4}x^2 \right) \mathrm{d}x = \pi \left[\frac{1}{2}x^2 - \frac{1}{12}x^3 \right]_0^4 = \frac{8\pi}{3} \text{ units}^3$$

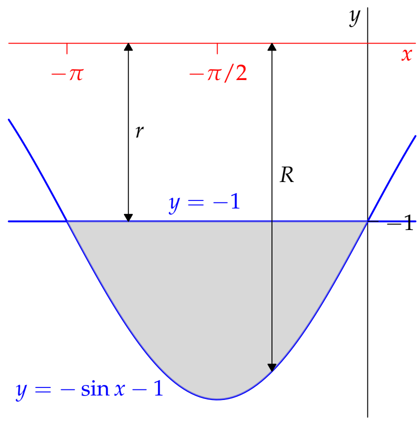


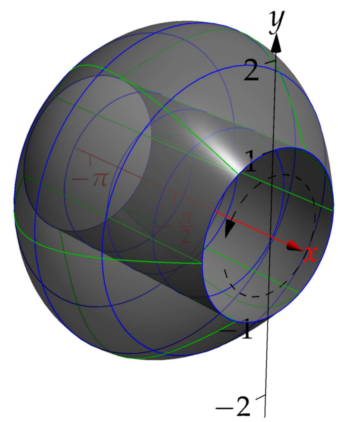


2. (Calculator) Find the volume of the solid generated when the region bounded by the curves $y = \sin x - 1$ and $y = -1$ on $[-\pi, 0]$ is revolved about the x-axis.

   The bounds are given as $x = \pi$ and $x = 0$. If we look at the graph of the curves, we observe that the 'outer' curve is the sine graph and the 'inner' curve is the straight line. The volume of the solid is then

$$V = \pi \int_{-\pi}^{0} \left((\sin x - 1)^2 - (-1)^2\right) \mathrm{d}x \approx 17.501 \text{ units}^3$$





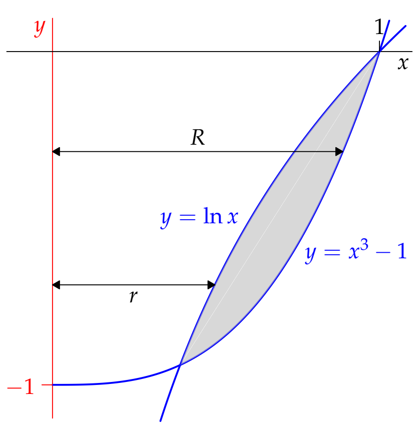
3. (Calculator) Write, but do not evaluate an integral expression which finds the volume of the solid generated when the region bounded by $y = \ln x$ and $y = x^3 - 1$ is rotated about the y-axis.

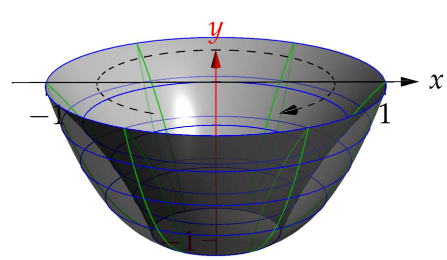
   Since we revolve around the y-axis, we must rewrite our equations as functions of y:

$$y = \ln x \iff x = e^y \quad \text{and} \quad y = x^3 - 1 \iff x = (y+1)^{1/3}$$

   Using our graphing calculators, we can find the y-values of intersection of these curves for our bounds: $e^y = (y+1)^{1/3} \implies y = -0.94, 0$; we let $A = -0.94$. We also notice from the graphs that the 'outer' curve is $x = (y+1)^{1/3}$ and our 'inner' $x = e^y$ so our integral expression which computes volume is

$$V = \pi \int_{A}^{0} \left((y+1)^{2/3} - e^{2y}\right) \mathrm{d}y$$





Revolution About Other Axes

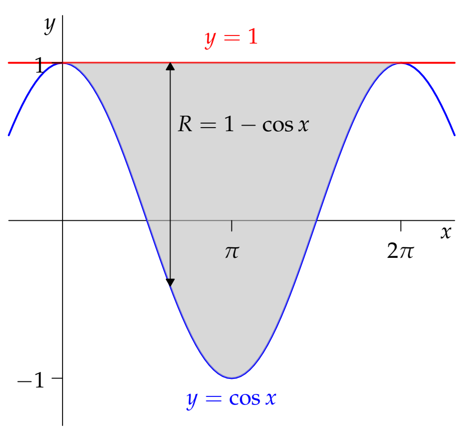
Notice that Theorems 5.22, 5.24 and 5.26 do not specify that the bounded regions must be rotated about the coordinate axes (x-axis and y-axis). As long as we are rotating about *any* horizontal or vertical axis, the approach still holds.

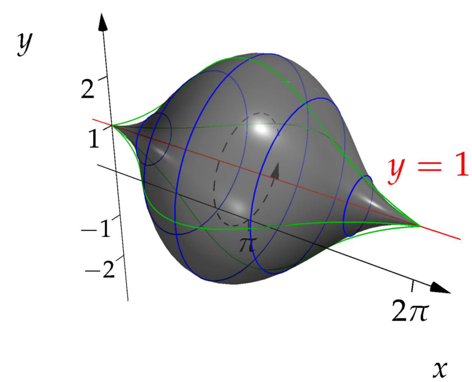
Simply keep in mind that R in the formulas represent the (horizontal or vertical) *distance* between the boundary of the area and the axis of revolution.

Examples 5.28. 1. Write, but do not evaluate an integral expression that computes the volume of the solid obtained by revolving the area enclosed by $y = \cos x$ and $y = 1$ over $0 \leq x \leq 2\pi$ around the horizontal line $y = 1$.

The axis of revolution is *horizontal*, so we must integrate with respect to x. One of the boundaries of the region is the axis of revolution, so we are using the *disc* method. The limits are also given, so all that remains is to find R, which is the horizontal distance between the curve $y = \cos x$ and $y = 1$ at any x-value. Hopefully it is clear that this distance is $R = 1 - \cos x$. So

$$V = \pi \int_0^{2\pi} (1 - \cos x)^2 \, dx \qquad (*)$$





Another useful way to think about revolutions about other axes is *translation*. If we translate *all* the curves downward such that the axis of revolution *becomes* the x-axis, then the volume of that solid will be identical to that of the solid which is revolved about the other line. In this example, if we imagine translating the curves downward

$$y = 1 - 1 = 0, \qquad y = \cos x - 1$$

and we rotate about the x-axis, the volume of the solid of revolution is

$$V = \pi \int_0^{2\pi} (\cos x - 1)^2 \, dx$$

which has the same value as the integral above in $(*)$!

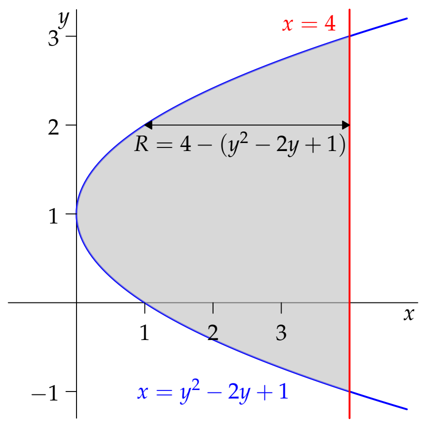
2. Suppose a region is bounded by the curve $x = y^2 - 2y + 1$ and the vertical line $x = 4$. Write, but do not evaluate, an integral expression that equals the volume of the solid generated when the given region is revolved around $x = 4$.

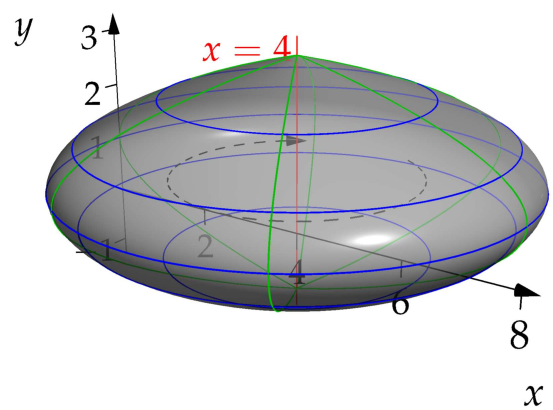
   The axis of revolution $x = 4$ is vertical, so we integrate with respect to y. We first need the y bounds, so we find the intersection between the parabola and the axis:

$$y^2 - 2y + 1 = 4 \Longrightarrow y^2 - 2y - 3 = (y+1)(y-3) = 0 \Longrightarrow y = -1, 3$$

   We use the disc method since the axis of revolution is one of the boundaries. By observation, the (horizontal) distance between the axis $x = 4$ and the curve $x = y^2 - 2y + 1$ is $4 - (y^2 - 2y + 1)$, so the volume is given by

$$V = \int_{-1}^{3} \left(4 - (y^2 - 2y + 1)\right)^2 \, dy$$





3. Write, but do not evaluate an integral expression which calculates the volume of the solid generated when the region bounded by the curves $y = 4 - 2x$ and $y = x^2 - 4x + 4$ are revolved around the line $x = -1$.

   The first thing to note is that we revolve the region about a vertical line, so we must write the equations of each curve as functions of y:

$$y = 4 - 2x \Longrightarrow x = 2 - \frac{1}{2}y$$
$$y = x^2 - 4x + 4 = (x-2)^2 \Longrightarrow x - 2 = \pm\sqrt{y} \Longrightarrow x = 2 \pm \sqrt{y}$$

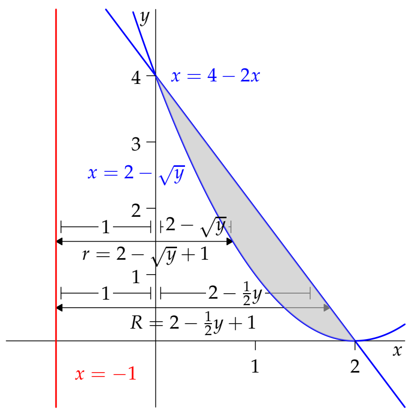
   where the negative square root represents the right half of the parabola and the positive the right half. To find the vertical bounds, we find the intersections:

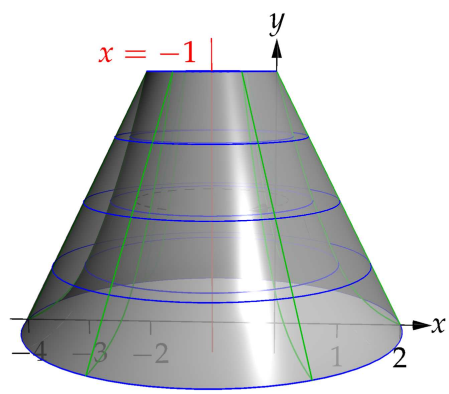
$$2 - \frac{1}{2}y = 2 + \sqrt{y} \Longrightarrow -\frac{1}{2}y = \sqrt{y} \Longrightarrow y = 0 \qquad \text{(right half)}$$
$$2 - \frac{1}{2}y = 2 - \sqrt{y} \Longrightarrow \frac{1}{2}y = \sqrt{y} \Longrightarrow y = 0, 4 \qquad \text{(left half)}$$

   So the parabola intersects the line at its vertex at $y = 0$ and at $y = 4$.

Now we determine that the line $x = 2 - \frac{1}{2}y$ is further away from the axis of revolution $x = -1$ than the parabola. Thus the distances $R = 2 - \frac{1}{2}y + 1$ and $r = 2 + \sqrt{y} + 1$, and so the integral which calculates volume is

$$V = \pi \int_0^4 \left(\left(2 - \frac{1}{2}y + 1\right)^2 - (2 + \sqrt{y} + 1)^2 \right) \mathrm{d}y$$





Volumes of Solids with Known Cross Sections

So far we've found volumes of solids whose cross sections are discs or washers. But our treatment would work for *any* solid, provided we know its cross-sectional areas.

Theorem 5.29. Suppose a 3-dimensional solid has cross sections whose areas are given by A, where A is a function of a either x or y. Then the volume of the solid over $a \leq x \leq b$ or $c \leq y \leq d$ is

$$V = \int_a^b A(x)\,\mathrm{d}x \quad \text{or} \quad V = \int_c^d A(y)\,\mathrm{d}y$$

This is the general method for finding volumes of solids with known cross sections. When doing volumes of solids of revolution, we simply had $A = \pi R^2$, where R was another function of x or y, since the cross sections were circles!

Here is the method:

1. We will be told if cross sections are particular shapes perpendicular to the x-axis or y-axis. If the former, we integrate with respect to x, and if the latter, we integrate with respect to y.

2. Find the (horizontal or vertical) bounds, as we did with area problems and solids of revolution earlier. Either they are given, or we must find them by finding intersections.

3. Derive the area formula A for the given cross section shapes in terms of the *base*.

4. Over the bounds we found, integrate the area formula A, replacing the base b with $f(x) - g(x)$ or $f(y) - g(y)$ (top minus bottom or right minus left).

Of course, you will never be required to draw a picture, but being able to visualize the solid in question is very helpful for finding the area formula of the cross-sectional shape!

5.4 Differential Equations

The final section weaves together elements of differential and integral calculus.

Definition 5.30. A *differential equation* is an equation with one or more terms which are the derivative(s) of one or more functions.

Suppose y is a function of x, so $y = y(x)$. Examples of differential equations for this function are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 4y^3 \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 0$$

Such equations not only appear in pure mathematics, but are also used to model and solve problems in *many* fields. A countless number of phenomena in science and economics can be modeled with differential equations.

In this course we only deal with differential equations with only one derivative term. These are known as *first-order ordinary*[28]differential equations.

For many problems, we are given a differential equation and must recover the original function y. This will always involve integration.

Example 5.31. Suppose we are given the very simple differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x$$

In this scenario, we can directly integrate to find y:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \Longrightarrow y = \int x\,\mathrm{d}x = \frac{1}{2}x^2 + C$$

Notice that in solving the differential equation we obtain a constant of integration C.

Definition 5.32. For an ordinary differential equation, a solution $y = f(x) + C$ is called a *general solution* to the differential equation, which is a family of curves which all satisfy the differential equation.

In contrast, if we are given *initial conditions* for the problem, i.e. a coordinate (x_0, y_0) , then we can evaluate C. This gives us a *particular solution* to the problem, which is one particular curve from the family of curves described by the general solution.

Example 5.31 cont. Suppose we are given the same differential equation along with initial condition

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x, \quad y(0) = 1$$

which corresponds to the point $(0, 1)$. To find the particular solution:

$$y = \frac{1}{2}x^2 + C \Longrightarrow 1 = \frac{1}{2}(0)^2 + C \Longrightarrow C = 1 \Longrightarrow y = \frac{1}{2}x^2 + 1$$

The *general solution* was $y = \frac{1}{2}x^2 + C$. The *particular solution* is $y = \frac{1}{2}x^2 + 1$.

[28]As opposed to *partial* differential equations.

Slope Fields

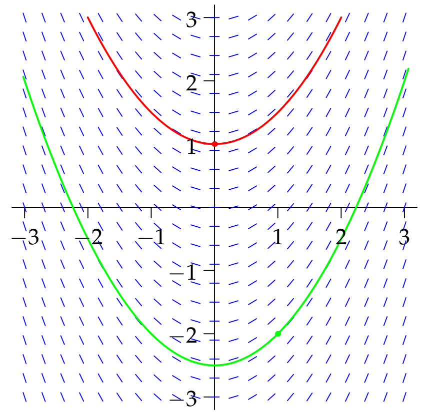
As we already made clear, we cannot find the particular solution to a differential equation unless we are given an initial condition. But given a differential equation, we can visualize *all* solutions.

Definition 5.33. A *slope field* is a graphical representation of the solutions to a first-order differential equation.

Usually we are not asked to draw a slope field: a computer does so for us. But if we are, we only need to draw at a handful of points.
And given any slope field, we may draw the curve of the particular solution through any initial condition point (x_0, y_0) by simply following the lines!

Example 5.31 cont. Suppose we have the differential equation

$$\frac{dy}{dx} = x$$

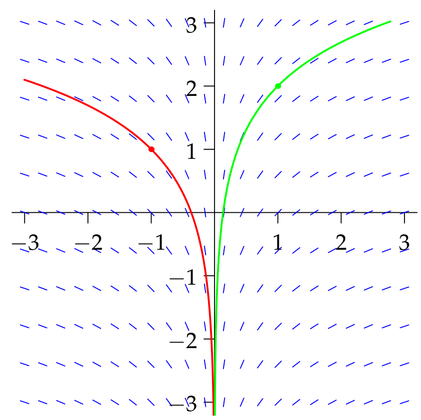


whose slope field is given on the right. Notice that the *slope* of each of the tiny lines is precisely given by substituting (x, y) into the differential equation.
The particular solution with initial condition $(0, 1)$ and the particular solution through $(1, -2)$ are drawn. Notice that the curves simply *follow the lines*.

Recall that if $f(x) = x^{-1}$, then the anti-derivatives of f are $F(x) = \ln|x| + C$ for a constant C. Does this mean that *all* anti-derivatives have the form $\ln|x| + C$? The answer is *no*!

Examples 5.34. 1. Consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x}$$



whose slope field is given alongside. The solutions curve through the points $(-1, 1)$ and $(1, 2)$ are sketched.

Notice that the value of $\frac{dy}{dx}$ is undefined whenever $x = 0$: the blue lines are therefore not drawn through the y-axis.

Strangely, the particular solutions with the aforementioned initial conditions have curves which do not simply differ by a constant. They point in totally different directions!

If we tried to find the general solution for the differential equation, we would get

$$\frac{dy}{dx} = \frac{1}{x} \Longrightarrow y = \int \frac{1}{x}\,dx = \begin{cases} \ln x + C_1, & x < 0 \\ \ln(-x) + C_2, & x > 0 \end{cases}$$

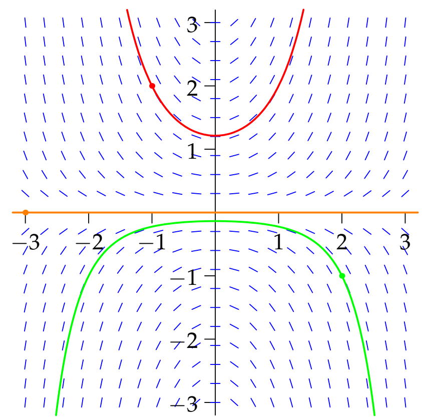
Because of the ugliness above, mathematicians will typically write $y = \ln|x| + C$.

2. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy$$

which is shown alongside. The solution curves with initial conditions $y(-1) = 2$, $y(2) = -1$, and $y(-3) = 0$ are also sketched.

Notice this time, as well, that the solutions differ completely based on the initial conditions. The solution curve through the origin seems to be a line while above and below the x-axis, the solutions are U-shaped curves opening in either direction.



Now that we're clear that a particular solution may be highly dependent on the initial condition, we may go about solving differential equations.

Solving Differential Equations

All of the differential equations we are meant to solve in this course are *separable* differential equations. The process is as follows:

1. Separate the variables in the differential equation such that the y and $\mathrm{d}y$ terms are on one side and the x and $\mathrm{d}x$ terms are on the other. This may involve factoring or using other algebraic properties of functions.
2. Indefinitely integrate both sides of the equation with respect to the respective variable. Add a constant of integration C to *only* one side of the equation to obtain the *general solution*.
3. Substitute the initial condition (x_0, y_0) into x and y to solve for C.
4. Isolate y to obtain the *particular solution* for the differential equation.

Examples 5.35. 1. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x^2+1}$$

Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(0) = -2$.

First, we separate variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x^2+1} \Longrightarrow \frac{1}{y}\,\mathrm{d}y = \frac{1}{x^2+1}\,\mathrm{d}x$$

Then we integrate to find the particular solution:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x^2+1}\,\mathrm{d}x \Longrightarrow \ln|y| = \arctan x + C$$

Then we use the initial condition $f(0) = -2 \leftrightsquigarrow (x, y) = (0, -2)$:

$$\ln|-2| = \arctan 0 + C \implies \ln 2 = C$$

from which we get $\ln|y| = \arctan x + \ln 2$. Then we isolate y to get our particular solution:

$$\ln|y| = \arctan x + \ln 2 \implies e^{\ln|y|} = |y| = e^{\arctan x + \ln 2} = e^{\arctan x} \cdot e^{\ln 2} = 2e^{\arctan x}$$

In order to isolate y, we must algebraically remove the absolute value. To do so, recall that we must multiply either side of the equation by ± 1:

$$y = \pm 2e^{\arctan x}$$

How can we choose the positive or negative case? We re-use the initial condition:

$$-2 = \pm 2e^{\arctan 0} = \pm 2 \implies y = -2e^{\arctan x}$$

whence we chose the negative case.

2. Consider the following differential equation:

$$\frac{dy}{dx} = 3x \sec y$$

Find the particular solution to the given differential equation with initial condition $y(1) = 0$.

First we separate variables:

$$\frac{dy}{dx} = 3x \sec y \implies \cos y \, dy = 3x \, dx$$

Then we can integrate both sides to get the general solution:

$$\int \cos y \, dy = \int 3x \, dx \implies \sin y = \frac{3}{2}x^2 + C$$

We find C by using our initial condition $y(1) = 0 \leftrightsquigarrow (x, y) = (1, 0)$:

$$\sin 0 = \frac{3}{2}(1)^2 + C \implies C = -\frac{3}{2} \implies \sin y = \frac{3}{2}x^2 - \frac{3}{2}$$

Then isolate y for the particular solution:

$$\sin y = \frac{3}{2}x^2 - \frac{3}{2} \implies y = \arcsin\left(\frac{3}{2}x^2 - \frac{3}{2}\right)$$

As you can see, solving differential equations often requires knowledge of algebra, trigonometry, and certainly requires the methods of integration discussed in a previous section.

Exponential Models

Recall that two variables a and b are called *directly proportional* or *vary directly* if, for some constant k, we have

$$a = kb$$

where k is called the *constant of proportionality* and we would often write $a \propto b$.

One specific real-life application of differential equations are those for which the rate of change of the quantity y with respect to time t is directly proportional to the quantity y. That is, for a constant k,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

If we attempt to find the general solution for this differential equation, we would separate variables then integrate, keeping in mind that k is *constant*:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky \implies \int \frac{1}{y}\,\mathrm{d}y = \int k\,\mathrm{d}t \implies \ln|y| = kt + C \implies |y| = e^{kt+C}$$

Since we use this differential equation to model real-life situations, it is always the case that the quantity y is positive, so we discard the negative case when removing the absolute value.

$$y = e^{kt+C} = e^C \cdot e^{kt} = Ae^{kt}$$

where we use the constant $A = e^C$ since C is constant $\implies e^C$ is constant.

Theorem 5.36. Suppose that the rate of change of the quantity y with respect to time t is directly proportional to the quantity y, i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

for a constant k. Then the general solution to the differential equation is

$$y = Ae^{kt}$$

A is the *initial value* of y and k is called the *growth constant* if $k > 0$ and the *decay constant* if $k < 0$.

A is the initial value since, when $t = 0$, $y(0) = Ae^0 = A$.

Examples 5.37. 1. (Calculator) Suppose a bacterial culture is grown in a lab dish such that the rate of change of the population P, in millions of bacteria, with respect to time t hours, is proportional to the population P at the time. At $t = 0$ hours, there are 2 million bacteria in the dish, and at $t = 3$ hours, there are 4 million bacteria in the dish.

(a) Find the population of bacteria after 7 hours.

(b) At what time t will the population of bacteria reach 5 million?

Sample response:

(a) Since the rate of change of P with respect to t is proportional to the population P, we have

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

for a constant k. If we follow our previous method for finding the general solution:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \implies \int \frac{1}{P}\,\mathrm{d}P = \int k\,\mathrm{d}t \implies P = Ae^{kt}$$

To find A, we use our initial condition $P(0) = 2 \implies A = 2$. So $P(t) = 2e^{kt}$. To find k, we use our *other* initial condition $P(3) = 4 \implies 4 = 2e^{3k} \implies k = \frac{1}{3}\ln 2 \approx 0.231$. Thus we have $P(t) = 2e^{0.231t}$. Since $P(t)$ represents the population of bacteria, in millions, at time t, we find $P(7) \approx 10.079$ million bacteria at $t = 7$ hours.

(b) We set $P = 5$ and find the corresponding time t:

$$5 = 2e^{0.231t} \implies t = \frac{\ln\frac{5}{2}}{0.231} \approx 3.966$$

So the population of bacteria reaches 5 million at about $t = 3.966$ hours.

2. (Calculator) The value C of a car t years after 2014 diminishes at a rate proportional to the value of the car. When the car is purchased in 2014, its value is \$29000. In 2016, the value of the car is \$21000.

(a) What is the value of the car in 2018 to the nearest dollar?

(b) In 2018, at what rate is the value of the car decreasing? Provide correct units.

Sample response:

(a) The wording suggests that we are working with the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} = kC \implies \int \frac{1}{C}\,\mathrm{d}C = \int k\,\mathrm{d}t \implies C = Ae^{kt}$$

for constants A and k. To find A, we use initial condition $C(0) = 29000 \implies A = 29000$, so $C(t) = 29000e^{kt}$. To find k, we use the other condition; in 2016, we are at time $t = 2$ so $C(2) = 21000 \implies 21000 = 29000e^{2k} \implies k \approx -0.161$. Therefore $C(t) = 29000e^{-0.161t}$ and the value of the car in 2018 is $C(4) \approx \$15207$ to the nearest dollar.

(b) We are asked for the rate of change of C in 2018, i.e. $t = 4$. We could easily evaluate $C'(4)$ to find this rate of change, or realize that the differential equation we wrote would give us the same answer. At $t = 4$, $C = 15207$ so

$$\left.\frac{\mathrm{d}C}{\mathrm{d}t}\right|_{C=15207} = -0.161 \cdot 15207 = -2454$$

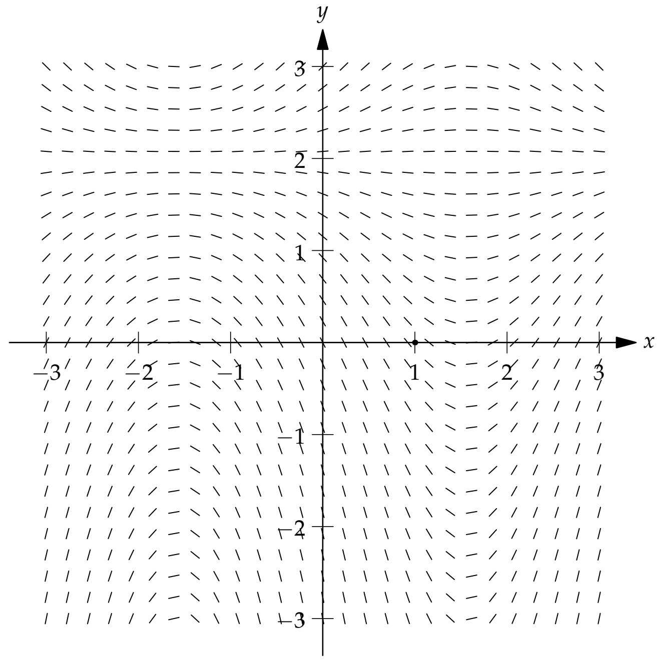
Therefore, in 2018, the rate at which the value of the car decreases is 2454 dollars per year. Evaluating $C'(4)$ would also have yielded this value, as you should check.

Exercises 5.4. 1. Consider the differential equation

$$\frac{dy}{dx} = (y-2)\cos x$$

Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 0$. The function f is defined for all real numbers.

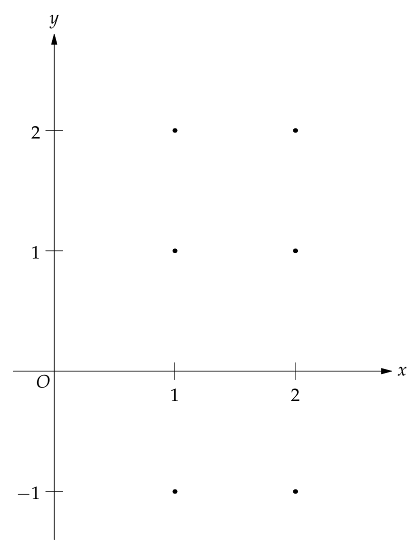
(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(1, 0)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(1.1)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(1) = 0$.

2. Consider the differential equation $\frac{dy}{dx} = ye^x$.



(a) On the axes provided on the right, sketch a slope field for the given differential equation at the six indicated points.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Using this, determine the concavity of all solution curves for the given differential equation in Quadrant IV. Explain your reasoning.

(c) Find $y = g(x)$, the particular solution to the differential equation with initial condition $g(3) = -2$.

3. (Hard) Consider the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{9-y^2}}{4-x}$$

Let $y = h(x)$ be the particular solution to the differential equation with the initial condition $h(2) = 0$.

   (a) Explain why the absolute maximum of h on the interval $-2 \leq x \leq 2$ must occur at $x = 2$.

   (b) Find $y = h(x)$, the particular solution to the given differential equation with initial condition $h(2) = 0$.

4. A mug of hot water is set down on a table to cool. The decreasing function W models the temperature of the water at time t, where W is measured in degrees Celsius (°C) and t is measured in minutes since the mug was set down. W satisfies the differential equation

$$\frac{dW}{dt} = \frac{1}{6}(20 - W)$$

At $t = 0$, the temperature of the water is 87°C. It is known that $W(t) > 20$ for all values of t.

   (a) Using correct units, explain the value $\lim_{t\to\infty} W(t) = 20$ in the context of the problem.

   (b) By considering the differential equation, explain why W does not attain a relative maximum or relative minimum for any time t.

   (c) Use separation of variables to find an expression for $W(t)$, the particular solution to the given differential equation with initial condition $W(0) = 87$.

5. (Calculator) The rate at which a certain chemical reaction proceeds at a certain temperature is directly proportional to the quantity of one reactant[29]. At one instant, 4.12 grams of substance A are present. 37 minutes later, the mass of unreacted A remaining is measured to be 0.39 grams.

   (a) Express this information as a differential equation and, using appropriate units, specify what your introduced variables represent.

   (b) Find the particular solution to this differential equation using the initial conditions detailed in the problem.

   (c) What quantity of substance A remains after 45 minutes?

   (d) Find the *half-life* of substance A, the period of time it takes for the quantity of the substance to reduce to half of its initial value.

[29] In case you're interested, in chemistry, such reactions are called *first-order reactions*. An example is:

$$2N_2O_5 \longrightarrow 4NO_2 + O_2$$

6. When funds are invested into an account that offers interest *compounded continuously*, the rate at which the amount of money in the account grows varies directly with the amount of money in the account at that moment.

   (a) By solving the differential equation, with constant annual interest rate r:

   $$\frac{\mathrm{d}A}{\mathrm{d}t} = rA$$

   Obtain the formula $A(t) = Pe^{rt}$, the amount $A(t)$ in an account after t years when interest is compounded continuously where $A(0) = P$ is the *principal* (or initial) investment.

   (b) If Bob deposits \$4225 into an account that earns 3.65% annual interest compounded continuously, what will be the balance in the account after 6 years?