Exercises 1.1 Solutions

1. As $x \to 2^-$, values of g seem to be decreasing infinitely, and as $x \to 2^+$, values of g seem to be increasing infinitely. So

$$\lim_{x\to 2^-} g(x) = -\infty \text{ and } \lim_{x\to 2^+} g(x) = \infty \Longrightarrow \lim_{x\to 2} g(x) = \text{DNE}$$

   The overall limit does not exist because the left- and right-side limits are different.

2. (a) $\displaystyle\lim_{x\to -1} \left(2x^2 - 15x\right) = 2(-1)^2 - 15(-1) = 2 + 15 = 17$

   (b) $\displaystyle\lim_{x\to 6} 23 = 23$; the limit of a constant function is simply the constant!

   (c) $\displaystyle\lim_{x\to -3} \frac{27 - x^3}{x^2 - 9} = \lim_{x\to -3} \frac{(3+x)(9-3x+x^2)}{(x+3)(x-3)} = \lim_{x\to -3} \frac{9 - 3x + x^2}{(x-3)} = \frac{9+9+9}{-6} = -\frac{9}{2}$

   In the factoring step, we used the *difference of cubes* formula.

   (d) $\displaystyle\lim_{x\to -2} \frac{x^3 + 5x^2 - 4x - 20}{3x^2 + 2x - 8} = \lim_{x\to -2} \frac{x^2(x+5) - 4(x+5)}{(3x-4)(x+2)} = \lim_{x\to -2} \frac{(x^2-4)(x+5)}{(3x-4)(x+2)}$

   $\displaystyle= \lim_{x\to -2} \frac{(x+2)(x-2)(x+5)}{(3x-4)(x+2)} = \lim_{x\to -2} \frac{(x-2)(x+5)}{3x-4} = \frac{(-2-2)(-2+5)}{3(-2)-4} = \frac{6}{5}$

   Note the use of *factoring by grouping*.

   (e) $\displaystyle\lim_{\theta\to \frac{\pi}{3}} \sec\theta = \sec\frac{\pi}{3} = 2$

   (f) $\displaystyle\lim_{x\to \frac{3}{2}} \lfloor x \rfloor = \left\lfloor \frac{3}{2} \right\rfloor = 1$; $x = 1.5$ is not at one of the jumps!

3. $\displaystyle f(x) = \frac{x-1}{(2x+1)(x-1)} = \frac{1}{2x+1},\ x \neq 1 \Longrightarrow f$ has a vertical asymptote at $x = -\frac{1}{2}$.

$$\lim_{x\to 1} \frac{x-1}{2x^2 - x - 1} = \lim_{x\to 1} \frac{1}{2x+1} = \frac{1}{3}$$

   Since $\displaystyle\lim_{x\to 1} f(x)$ is finite, $x = 1$ is *not* a vertical asymptote of f.

4. If we view the graph of $y = \cot\theta$, we would see that

$$\lim_{\theta\to \pi^-} \cot\theta = -\infty \quad \text{and} \quad \lim_{\theta\to \pi^+} \cot\theta = \infty$$

   From this, we may deduce that $\cot\theta$ has a vertical asymptote at $\theta = \pi$.

5. (a) This is an exercise to remind you of the graphs of the trigonometric functions. The graph of $g(x) = 3\csc 4(x-1) - 5$ has a translation of 1 unit to the right, followed by a horizontal shrink by a factor of $\frac{1}{4}$, followed by a vertical stretch by a factor of 3, followed by a translation of 5 units down of the parent function $f(x) = \csc x$. The only transformations which affect the location of the vertical asymptotes are the horizontal dilation and translation. If you use a graphing utility, we can find

$$\lim_{x\to 1^-} g(x) = -\infty \quad \text{and} \quad \lim_{x\to 1^+} g(x) = \infty \Longrightarrow \lim_{x\to 1} g(x) = \text{DNE}$$

   Though the limit does not exist, since either of the one-sided limits at $x = 1$ are infinity, g has a vertical asymptote at $x = 1$.

(b) More generally, as previously said, only the constants b and h affect the location of the vertical asymptotes of cosecant. Recall that the *period* of cosecant is

$$\text{Period} = \frac{2\pi}{b}$$

The asymptotes of the function repeat periodically, and the parent function has an asymptote at $x = 0$. So, taking into account the horizontal translation as well, we have that the vertical asymptotes of $y = a \csc b(x - h) + k$ occur at $x = h + \frac{2\pi}{b} n$ for any integer n.

6. (a) $\lim_{x\to\infty} \frac{e^{-x}}{e^x} = \lim_{x\to\infty} \frac{1}{e^{2x}} = 0$

   (b) $\lim_{x\to-\infty} \frac{e^{-x}}{e^x} = \lim_{x\to-\infty} \frac{1}{e^{2x}} = 0$

   (c) The natural log function increases unboundedly as $x \to \infty$. Due to the negative, the sign flips. So

   $$\lim_{x\to\infty} (5 - \ln(2x - 1)) = -\infty$$

   (d) $\lim_{x\to\infty} \frac{x^{7/4} + 3x^2 - 10x^{5/2}}{7x + 13x^{8/3}} = \lim_{x\to\infty} \frac{x^{7/4}}{13x^{8/3}} = \lim_{x\to\infty} \frac{1}{13x^{11/12}} = 0$

   (e) $\lim_{x\to-\infty} \frac{1 - x + 12x^2}{5x^2 + 3 + 10x} = \lim_{x\to-\infty} \frac{12x^2}{5x^2} = \lim_{x\to-\infty} \frac{12}{5} = \frac{12}{5}$

   (f) $\lim_{x\to\infty} \frac{-5x^3 + 3x - 9}{6x^2 + 19} = \lim_{x\to\infty} \frac{-5x^3}{6x^2} = \lim_{x\to\infty} \frac{-5x}{6} = -5(\infty) = -\infty$

7. To find the vertical asymptotes, we simplify first then find the zeros of the denominator:

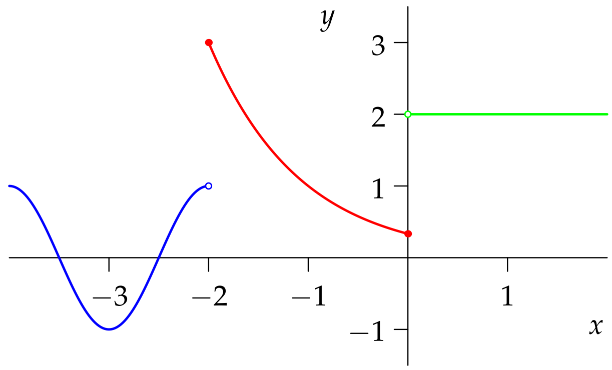
   $$h(x) = \frac{10x^2 - 29x - 21}{2x^2 - x - 15} = \frac{(5x + 3)(2x - 7)}{(2x + 5)(x - 3)}$$

   whence the vertical asymptotes are $x = -\frac{5}{2}$ and $x = 3$. The horizontal asymptote depends on the limits at infinity:

   $$\lim_{x\to\infty} \frac{10x^2 - 29x - 21}{2x^2 - x - 15} = \lim_{x\to\infty} \frac{10x^2}{2x^2} = \lim_{x\to\infty} \frac{10}{2} = 5$$

   The limit at negative infinity is identical. So the horizontal asymptote of h is $y = 5$.

8. The graph of $y = f(x)$ is shown below:



(a) $\lim_{x \to -2^-} f(x) = 1$

(b) $\lim_{x \to -2^+} f(x) = 3$

(c) $\lim_{x \to -2} f(x) = \text{DNE}$

(d) $\lim_{x \to 0^-} f(x) = \frac{1}{3}$

(e) $\lim_{x \to 0^-} f(x) = 2$

(f) $\lim_{x \to 0} f(x) = \text{DNE}$

Exercises 1.2 Solutions

1. For $f(x) = \sqrt{9 - x^2}$ at $x = -2$:

   $f(-2)$ is defined.
   $\lim_{x\to -2} f(x)$ exists.
   $f(-2) = \lim_{x\to -2} f(x) = \sqrt{5}$

   Therefore f is continuous at $x = -2$.

2. (a) The range of $\cos\left(\frac{1}{x^2}\right)$ is $-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$.
   (b) We use the Squeeze Theorem and our answer in part (a):

   $$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1 \Longrightarrow -x^2 \le x^2 \cos\left(\frac{1}{x^2}\right) \le x^2$$

   The inequality is preserved because $x^2 \ge 0$ for all values of x. Thus

   $$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0 \Longrightarrow \lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$$

3. Each limit requires some algebraic manipulation combined with the limit laws. For brevity, we use some results if they have already been proven.

   (a) $\lim_{x\to 0} \frac{\sin^2 x}{3x} = \lim_{x\to 0} \frac{\sin x}{3x} \cdot \lim_{x\to 0} \sin x = \frac{1}{3} \cdot 0 = 0$
   (b) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x \cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot 1 = 1$
   (c) $\lim_{x\to 0} \frac{\sin 4x}{9x} = \frac{4}{9}$
   (d) $\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{x \cos x}{\sin x} = \lim_{x\to 0} \cos x \cdot \frac{x}{\sin x} = 1 \cdot 1 = 1$
   (e) $\lim_{x\to 0} \frac{x^2 + x}{\sin x} = \lim_{x\to 0} \left(\frac{x^2}{\sin x} + \frac{x}{\sin x}\right) = \lim_{x\to 0} \left(x \cdot \frac{x}{\sin x} + \frac{x}{\sin x}\right) = 0 + 1 = 1$
   (f) $\lim_{x\to 0} 2x \csc 3x = \lim_{x\to 0} \frac{2x}{\sin 3x} = \frac{2}{3} \lim_{x\to 0} \frac{3x}{\sin 3x} = \frac{2}{3}$

4. $\lim_{x\to 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x\to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{1 + \cos x} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$

5. For a rational function, we may possibly have infinite discontinuities at vertical asymptotes, or removable disctontinuities at the location of holes.

   $$h(x) = \frac{x^3 - 125}{2x^2 - 3x - 35} = \frac{(x-5)(x^2 + 5x + 25)}{(2x+7)(x-5)} = \frac{x^2 + 5x + 25}{2x + 7}, \quad x \ne 5$$

   So h has an *infinite discontinuity* at $x = -\frac{7}{2}$ and a removable discontinuity at $x = 5$. To remove the latter, we set

   $$\tilde{h}(5) = \lim_{x\to 5} h(x) = \lim_{x\to 5} \frac{x^2 + 5x + 25}{2x + 7} = \frac{75}{17}$$

6. In order for f to be continuous everywhere, we must ensure there are no discontinuities. The only possible locations of discontinuity are at the breakpoints $x = 1$ and $x = 3$. If we refer to the three-part definition, all three conditions may be fulfilled by making the left- and right-side limits equivalent at both x-values:

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) \quad \text{and} \quad \lim_{x\to 3^-} g(x) = \lim_{x\to 3^+} g(x)$$
$$\implies \begin{cases} \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(ax+b) \\ \lim_{x\to 3}(ax+b) = \lim_{x\to 3}(3^x - 10) \end{cases} \implies \begin{cases} a+b=2 \\ 3a+b=-1 \end{cases}$$

This is now a system of two linear equations for unknown variables a and b, which can be solved via whichever method you have learned previously (elimination or substitution). However we solve, we get the solutions $a = -\frac{3}{2}$ and $b = \frac{7}{2}$.

7. Function f is a combination of many continuous functions, so it is continuous wherever it is defined. So we simply find the domain by identifying any possible intervals or values for which f is undefined. The term $(x+4)^{5/2}$ is undefined whenever $x + 4 < 0 \implies x < -4$. We have a denominator $x - 3$ as well, which is undefined when $x = 3$. Thus f is continuous on the intervals $(-4, 3)$ and $(3, \infty)$.

8. W is continuous and thus satisfies the hypotheses of IVT. Between $t = 0$ and $t = 1.3$, there must be a time t for which the water level $W(t) = 75\,\text{m}$ exactly. We can similarly argue between $t = 2.7$ and $t = 4.4$, as well as $t = 5.9$ and $t = 7.5$. So the minimum number of times the depth of the water reservoir was exactly 75 m is 3.

9. (a) Using a calculator, $f(-1) = -1.557$ and $f(0) = 1$.
   (b) f is a combination of continuous functions, so it is continuous. According to part (a) and IVT, there must be an x-value between $x = -1$ and $x = 0$ such that $f(x) = 0$.
   (c) $f(x) = 0$ at approximately $x = -0.638$

10. No: g is discontinuous at $x = 0$, which means it fails the continuity condition for IVT.

11. (a) Since g and h are continuous with $g(3) = h(3) = 5$, we have

$$\lim_{x\to 3} g(x) = \lim_{x\to 3} h(x) = 5. \qquad (*)$$

To test for continuity of k at $x = 3$, we refer to the three-part definition:

i. Since $g(x) \le k(x) \le h(x)$ on interval $(2, 4)$, we have

$$g(3) \le k(3) \le h(3) \implies 5 \le k(3) \le 5 \implies k(3) = 5$$

where $k(3)$ is certainly defined.

ii. Given the inequality $g(x) \le k(x) \le h(x)$ on interval $2 < x < 4$, we can use Squeeze Theorem and $(*)$ to show that

$$\lim_{x\to 3} k(x) \text{ exists and } \lim_{x\to 3} k(x) = \lim_{x\to 3} g(x) = \lim_{x\to 3} h(x) = 5$$

iii. From the previous two steps, the function value and limit are certainly the same.

Thus k is continuous at $x = 3$.

(b) We once again refer to the three-part defintion:

$$f(3) = 5(3)^2 \cdot h(3) - \frac{1}{25 - (k(3))^2} = 45 \cdot 5 - \frac{1}{25 - 25}$$

which is undefined. So f is not continuous at $x = 3$.