Exercises 2.1 Solutions

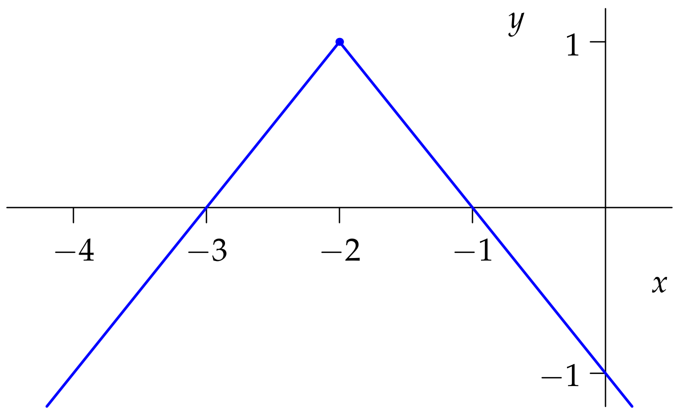
1. We can rewrite the function as

$$y = \begin{cases} 1+(x+2), & x<-2 \\ 1-(x+2), & x \geq -2 \end{cases} = \begin{cases} 3+x, & x<-2 \\ -1-x, & x \geq -2 \end{cases}$$

   To verify it is non-differentiable at $x=-2$, we must observe the following limits:

$$\lim_{h\to 0^-} \frac{3+x+h-(3+x)}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1$$
$$\lim_{h\to 0^+} \frac{-1-x-h-(-1-x)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

   Thus the derivative of the given function at $x=-2$ does not exist. Indeed, if we look at the graph of the curve, there is a *corner* at $x=-2$.



   The function is, however, continuous at $x=-2$. You can use the three-part definition to verify.

2. (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \dfrac{15-15}{h} = 0$

   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \dfrac{6(x+h)-6x}{h} = \lim_{h\to 0} \dfrac{6h}{h} = 6$

   (c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \dfrac{7-2(x+h)-7+2x}{h} = \lim_{h\to 0} \dfrac{-2h}{h} = -2$

3. (a) $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = 0$

   (b) $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = 6$

   (c) $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = -2$

   Quite boring...

4. Instead of finding $g'(x)$, we simply observe that $g(6)$ and $g'(6)$ are undefined, and thus g is not continuous at those x-values. Therefore g is also not differentiable at $x = 6$ and $x = -6$.

5. The slope of the tangent line to f at $x = 1$ is $f'(1)$:

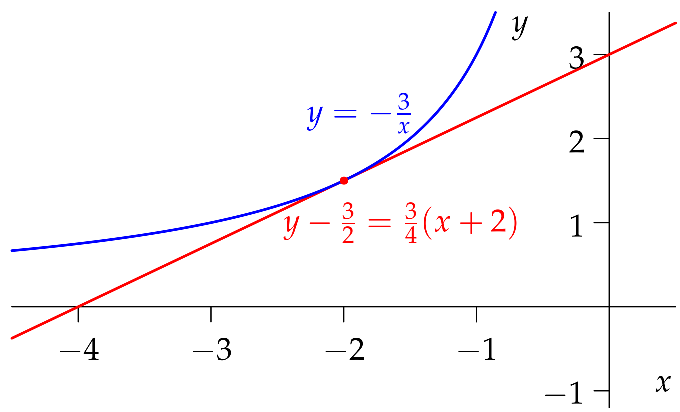
$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{-4(1+h)^2 + 4(1)^2}{h}$$
$$= \lim_{h\to 0} \frac{-4h^2 - 8h - 4 + 4}{h} = \lim_{h\to 0}(-4h - 8) = -8$$

6. At $x = -2$, we have $y = \frac{3}{2}$. To find the slope of the tangent, we need the derivative at $x = -2$:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-2} = \lim_{h\to 0} \frac{y(-2+h) - y(-2)}{h} = \lim_{h\to 0} \frac{-\frac{3}{2+h} + \frac{3}{2}}{h} = \lim_{h\to 0} \frac{-\frac{-3(2)+3(2+h)}{2(2+h)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3h}{4+2h}}{h} = \lim_{h\to 0} \frac{3}{4+2h} = \frac{3}{4}$$

Therefore the equation of the tangent line is

$$y - \frac{3}{2} = \frac{3}{4}(x+2)$$



7. If $y = x^3$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{3x^2 + 6xh + h^2 - 3x^2}{h} = \lim_{h\to 0}(6x + h) = 6x$$

Exercises 2.2 Solutions

1. Since $\tan x = \frac{\sin x}{\cos x}$, let $u = \sin x \Longrightarrow u' = \cos x$ and $v = \cos x \Longrightarrow v' = -\sin x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Since $\cot x = \frac{\cos x}{\sin x}$, let $u = \cos x \Longrightarrow u' = -\sin x$ and $v = \sin x \Longrightarrow v' = \cos x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

Since $\csc x = \frac{1}{\sin x}$, let $u = 1 \Longrightarrow u' = 0$ and $v = \sin x \Longrightarrow v' = \cos x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

2. (a) $y(2) = 5 - 2 + 2(2)^2 = 11$

   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1 + 4x \Longrightarrow \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} = -1 + 4(2) = 7$

   (c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4 \Longrightarrow \left.\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=2} = 4$

3. (a) Let $u = \sin x \Longrightarrow u' = \cos x$ and $v = \ln x \Longrightarrow v' = \frac{1}{x}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x \ln x = \cos x \ln x + \sin x \cdot \frac{1}{x} \cos x \ln x + \frac{\sin x}{x}$$

   (b) Let $u = 2e^x \Longrightarrow u' = 2e^x$ and $v = \tan x \Longrightarrow v' = \sec^2 x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}2e^x \tan x = 2e^x \tan x + 2e^x \sec^2 x = 2e^x(\tan x + \sec^2 x)$$

   (c) We can rewrite $\cos^2 x = \cos x \cdot \cos x$. Let $u = v = \cos x \Longrightarrow u' = v' = -\sin x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos^2 x = \cos x \cdot -\sin x + \cos x \cdot -\sin x = -2\sin x \cos x$$

4. A horizontal line has a slope of 0. So we find all points on the curve such that the derivative of y with respect to x is equal to 0.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 18x + 15 = 3(x^2 + 6x + 5) = 3(x+5)(x+1) = 0 \Longrightarrow x = -5, x = -1$$

To now find the corresponding y-coordinates of the points on the curve, we substitute these x-values into the original equation:

$$y(-5) = 23 \leftrightsquigarrow (-5, 23), \qquad y(-1) = -9 \leftrightsquigarrow (-1, -9)$$

5. First, $g'(x) = 6x^2 + a$. The tangent line has a slope of 9 at $x = 2$, meaning

$$g'(2) = 9 = 6(2)^2 + a \implies a = -15$$

Also, we know the point $(2, 8)$ lies on the graph of g, so

$$g(2) = 8 = 2(2)^3 - 15(2) + b \implies b = 22$$

6. (a) Let $u = \sqrt{t} = t^{1/2} \implies u' = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$ and $v = t^2 - 3 \implies v' = 2t$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\frac{1}{2\sqrt{t}} \cdot (t^2 - 3) - \sqrt{t} \cdot 2t}{(t^2 - 3)^2} = \frac{\frac{t^2-3}{2\sqrt{t}} - 2t\sqrt{t}}{(t^2 - 3)^2} = \frac{-3(t^2 + 1)}{2\sqrt{t}(t^2 - 3)^2}$$

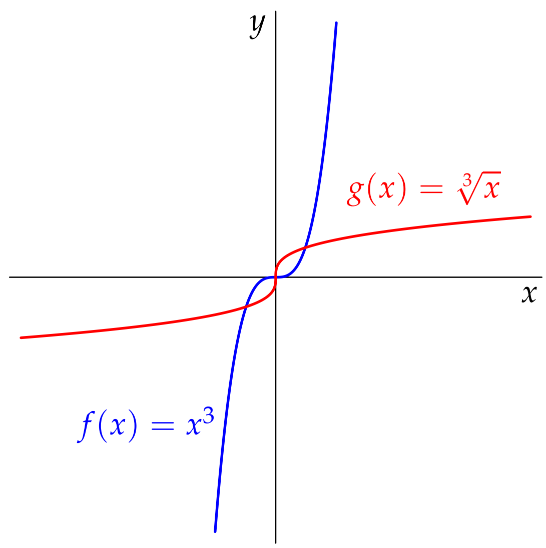
(b) If you're clever, you may notice we can break up the fraction!

$$y = \frac{t^3 - t}{t^2} = \frac{t^3}{t^2} - \frac{t}{t^2} = t - t^{-1}$$

from which we may simply use the power rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + t^{-2} = 1 + \frac{1}{t^2}$$

7. (a) The functions f and g are *inverses*. Indeed $f(g(x)) = g(f(x)) = x$.



(b) $f'(x) = 3x^2$, and $g(x) = x^{1/3} \implies g'(x) = \frac{1}{3}x^{-2/3}$

(c) $f'(2) = 12$ and $g'(8) = \frac{1}{12}$. This may be unsurprising, since one of the properties of inverse functions is that, visually, they are reflections over the line $y = x$.

(d) $f(2) = 8$, so the tangent line equation is $y - 8 = 12(x - 2)$.
$g(8) = 2$, so the tangent line equation is $y - 2 = \frac{1}{12}(x - 8)$.

8. If h is to be differentiable everywhere, it must be both continuous and differentiable at $x = 1$. To ensure that it is continuous, we need

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^+} h(x) \implies a + b(1) = 1^2 \implies a + b = 1$$

To ensure that it is differentiable, we need

$$\lim_{x\to 1^-} h'(x) = \lim_{x\to 1^+} h'(x) \implies b = 2(1) \implies b = 2 \implies a = -1$$

9. (a) Using the quotient rule, let $u = 1 \implies u' = 0$ and $v = x^3 \implies v' = 3x^2$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{0\cdot x^3 - 1\cdot 3x^2}{(x^3)^2} = \frac{-3x^2}{x^6} = -\frac{3}{x^4}$$

   (b) Trivially, using the power rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^{-4} = -\frac{3}{x^4}$$

   Using the power rule is obviously much easier!

10. (a) $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and $f^{(4)}(x) = \sin x$. The functions repeat every fourth iteration of differentiation!
   (b) $f^{(100)}(x) = -\sin x$

11. If $f(x) = c = cx^0$, then by the power rule, we have

$$f'(x) = 0\cdot cx^{-1} = 0$$

12. The proof can be followed by reading, but it includes a lot of algebra!

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Currently, there is not much we can do; we will start by subtracting and adding the term to the numerator $f(x+h)g(x)$:

$$\begin{aligned}
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x))}{h} + \lim_{h\to 0} \frac{g(x)(f(x+h) - f(x))}{h} \\
&= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Exercises 2.3 Solutions

1. (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(-\ln(\cos x)) = -\dfrac{1}{\cos x} \cdot -\sin x = \tan x$

   (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\csc(-4x) = -\csc(-4x)\cot(-4x) \cdot -4 = 4\csc(-4x)\cot(-4x)$

   (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}12(3x-7x^3+14)^5 = 60(3x-7x^3+14)^4(3-21x^2)$

   (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}e^{\sin x} = e^{\sin x} \cdot \cos x$

   (e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}(2x) = \dfrac{-1}{\sqrt{1-(2x)^2}} \cdot 2 = \dfrac{-2}{\sqrt{1-4x^2}}$

   (f) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2x^3-x^2)^{1/3} = \dfrac{1}{3}(2x^3-x^2)^{-2/3} \cdot (6x^2-2x) = \dfrac{6x^2-2x}{3\sqrt[3]{(2x^3-x^2)^2}}$

   (g) Let $u = x \Longrightarrow u' = 1$ and $v = \sqrt{\sec x} = (\sec x)^{1/2} \Longrightarrow v' = \frac{1}{2}(\sec x)^{-1/2} \cdot \sec x \tan x$:

   $$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{\sqrt{\sec x}} = \frac{1 \cdot (\sec x)^{1/2} - \frac{1}{2}x(\sec x)^{-1/2}\sec x \tan x}{\sec x}$$

   (h) When we differentiate, we need to multiply by the derivative of the argument of the natural logarithm. For that, let $u = (x+2)^3 \Longrightarrow u' = 3(x+2)^2$ and $v = x \Longrightarrow v' = 1$:

   $$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{(x+2)^3}{x}\right) = \frac{x}{(x+2)^3} \cdot \frac{3x(x+2)^2-(x+2)^3}{x^2} = \frac{2x-2}{x^2+2x}$$

   (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos\left(e^{x^5-3x}\right) = -\sin\left(e^{x^5-3x}\right) \cdot e^{x^5-3x} \cdot (5x^4-3)$

2. A vertical line has a slope which is undefined. So we are looking for the points on the curve for which the derivative of y with respect to x is undefined:

   $$\begin{aligned} &2(x^2+y^2)\cdot\left(2x+2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2x-2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Longrightarrow 4x^3+4x^2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+4xy^2+4y^3\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 2x-2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(4x^2y+4y^3+2y) = 2x-4xy^2-4x^3 \\ &\Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-4xy^2-4x^3}{4x^2y+4y^3+2y} \end{aligned}$$

   The derivative is undefined when the denominator is 0 $\iff$ $y = 0$ and $x \neq 0$. If we want the coordinates of these points, we substitute $y = 0$ into the equation of the curve:

   $$(x^2+0^2)^2 = x^2-0^2 \Longrightarrow x^4 = x^2 \Longrightarrow x = \pm 1$$

   The points on the curve which have a vertical tangent are $(-1,0)$ and $(1,0)$.

3. $f'(x) = \cos(\sin(\sin(\sin x))) \cdot \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$

4. (a) If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$ and $h'(2) = f'(g(2)) \cdot g'(2) = f'(-1) \cdot 3$ $= -4 \cdot 3 = -12$

   (b) If $k(x) = f(x)g(x)$, then $k'(x) = f'(x)g(x) + f(x)g'(x)$ and $k'(5) = f'(5)g(5) + f(5)g'(5)$ $= 12 \cdot 3 - 3 \cdot 10 = 6$

5. If $y = (1 - \frac{1}{3}x)^3$, then

$$\frac{dy}{dx} = 3\left(1 - \frac{1}{3}x\right)^2 \cdot -\frac{1}{3} = -\left(1 - \frac{1}{3}x\right)^2$$
$$\implies \frac{d^2y}{dx^2} = -2\left(1 - \frac{1}{3}x\right) \cdot -\frac{1}{3} = \frac{2}{3} - \frac{2}{9}x$$
$$\implies \frac{d^3y}{dx^3} = -\frac{2}{9}$$

6. First, notice that $f(1) = 3 \iff f^{-1}(3) = 1$, and that $f'(x) = 3x^2 - 4x + 5 \implies f'(1) = 4$. Now

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \implies (f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{4}$$

7. (a) Using implicit differentiation:

$$6x + 2 \cdot \frac{dy}{dx} = 2y + 2x \cdot \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{2y - 6x}{2 - 2x}$$

   We can also find the rate of change of x with respect to y:

$$6x \cdot \frac{dx}{dy} + 2 = 2x + 2y \cdot \frac{dx}{dy} \implies \frac{dx}{dy} = \frac{2 - 2x}{2y - 6x}$$

   (b) Unsurprisingly, the rates of change are reciprocal!

8. (a) Using implicit differentiation:

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$
$$\implies \frac{d^2y}{dx^2} = -\frac{y - x \cdot \frac{dy}{dx}}{y^2} = -\frac{y + x \cdot \frac{x}{y}}{y^2} = -\frac{x^2 + y^2}{y^3}$$

   (b) Again, using implicit differentiation:

$$2x - 2y \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{x}{y}$$
$$\implies \frac{d^2y}{dx^2} = \frac{y - x \cdot \frac{dy}{dx}}{y^2} = \frac{y - x \cdot \frac{x}{y}}{y^2} = \frac{y^2 - x^2}{y^3}$$

(c) Isn't this fun...

$$3x^2 + 2y + 2x \cdot \frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = \frac{-3x^2 - 2y}{2x}$$
$$\Longrightarrow \frac{d^2y}{dx^2} = \frac{(-6x - 2 \cdot \frac{dy}{dx}) \cdot 2x - (-3x^2 - 2y) \cdot 2}{(2x)^2} = \frac{2y}{x^2}$$

9. Recall the change-of-base formula:

$$y = \log_2 x = \frac{\ln x}{\ln 2} \Longrightarrow \frac{dy}{dx} = \frac{1}{x \ln 2}$$

10. At $x = 4$, $y(4) = 5^{\sqrt{4}} = 5^2 = 25$. The slope of the tangent is the derivative at $x = 4$:

$$\frac{dy}{dx} = 5^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot \ln 5 \Longrightarrow \left.\frac{dy}{dx}\right|_{x=4} = 25 \cdot \frac{1}{4} \cdot \ln 5 = \frac{25}{4} \ln 5$$

Thus the equations of the required tangent and normal line are, respectively

$$y - 25 = \frac{25}{4} \ln 5(x - 4) \qquad y - 25 = \frac{4}{25 \ln 5}(x - 4)$$

11. We have

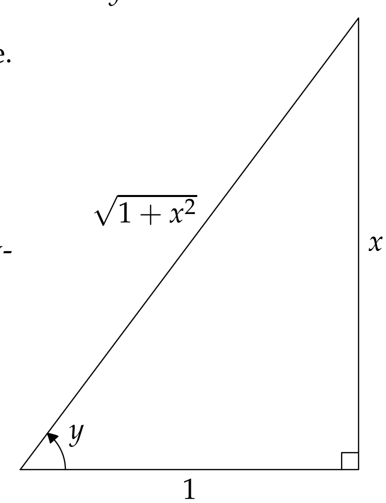
$$y = \arctan x \iff x = \tan y \Longrightarrow \frac{dx}{dy} = \sec^2 y \Longrightarrow \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sec^2 y}$$

We can get this expression in terms of x by viewing y as an angle. Since $x = \tan y$, we draw a right triangle using

$$\tan y = \frac{x}{1} = \frac{\text{opposite}}{\text{adjacent}}$$

The Pythagorean Theorem tells us that the length of the hypotenuse is $\sqrt{1 + x^2}$, and we then have

$$\sec y = \frac{\sqrt{1 + x^2}}{1} \Longrightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + x^2}$$



12. It is unnecessary to evaluate the limit. Notice that, for each of these questions, we are simply being asked for the derivatives of certain functions!

(a) We need the derivative of $f(x) = \sec(2x) \Longrightarrow f'(x) = 2 \sec(2x) \tan(2x)$

(b) Here, we want the derivative of $f(x) = \ln x$ evaluated at $x = e$:

$$f(x) = \ln x \Longrightarrow f'(x) = \frac{1}{x} \Longrightarrow f'(e) = \frac{1}{e}$$

13. First, we find

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$

Therefore the slope of the normal line for all points on the unit circle is $-\frac{y}{x}$. Let (x_0, y_0) be any point on the unit circle. Any normal line has the form

$$y - y_0 = -\frac{y_0}{x_0}(x - x_0) \implies y = y_0 - xy_0 - y_0 = xy_0$$

which has a y-intercept of 0 for any values of (x_0, y_0).

14. We can write $\frac{f(x)}{g(x)} = f(x) \cdot (g(x))^{-1}$. The power rule, product rule, and chain rule give us

$$\begin{aligned} \frac{d}{dx}\frac{f(x)}{g(x)} &= f'(x) \cdot (g(x))^{-1} + f(x) \cdot -(g(x))^{-2} \cdot g'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \end{aligned}$$