Exercises 3.1 Solutions

1. We check the conditions first:

   f is a power function and is defined for all x on $[-1,1]$. So f is continuous on that interval.

   By differentiating, we get $f'(x) = -3x(1+3x^2)^{-3/2}$, which is defined for all x on $(-1,1)$. So f is differentiable on this interval.

   $f(-1) = f(1) = \frac{1}{2}$

   Thus f satisfies the hypotheses of Rolle's Theorem on $[-1,1]$. We have

   $$f'(c) = \frac{-3c}{(\sqrt{1+3c^2})^3} = 0 \Longrightarrow c = 0$$

2. (a) Check the conditions for MVT:

   f is continuous on $[-2,3]$ since it is a polynomial.

   f' is also a polynomial, so f is differentiable on $(-2,3)$.

   The average rate of change of f from $x=-2$ to $x=3$ is

   $$\frac{f(3)-f(-2)}{3-(-2)} = \frac{0-5}{5} = -1$$

   Now

   $$f'(c) = -2c = -1 \Longrightarrow c = \frac{1}{2}$$

   (b) Check the conditions for MVT:

   g is continuous on $[0,3\pi]$.

   $g'(x) = 1+\cos x$ is defined on $(0,3\pi)$, so g is differentiable on that interval.

   The average rate of change of g from $x=0$ to $x=3\pi$ is

   $$\frac{g(3\pi)-g(0)}{3\pi-0} = \frac{3\pi}{3\pi} = 1$$

   Now

   $$g'(c) = 1+\cos c = 1 \Longrightarrow \cos c = 0 \Longrightarrow c = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

   (c) Check the conditions for MVT:

   h is defined and therefore continuous on $[-8,8]$.

   $h'(x) = \frac{2}{3}x^{-1/3}$ is undefined at $x=0$, which is in the interval $(-8,8)$. So h is *not* differentiable on $(-8,8)$.

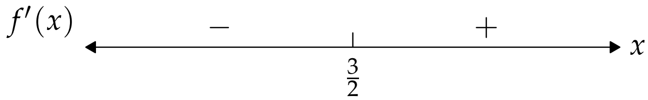
   Mean Value Theorem does not apply to h on $[-8,8]$.

3. The function $h(x) = x^{2/3}$ in part (c) of the previous question is one such example!

4. (a) For $f(x) = e^{x^2-3x}$, we first find all critical values:

$$f'(x) = e^{x^2-3x} \cdot (2x-3) \Longrightarrow x = \frac{3}{2}$$

We only have one critical value, which we can label on our first derivative sign chart:



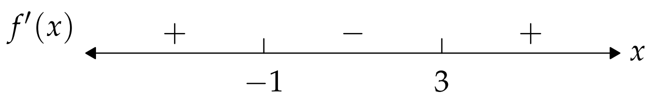
Where we get the signs from evaluating, for example, $f'(0)$ and $f'(2)$. Thus:

f is increasing on $(-\infty, \frac{3}{2})$ since f' is negative on that interval.
f is decreasing on $(\frac{3}{2}, \infty)$ since f' is positive on that interval.
f has a local minimum at $x = \frac{3}{2}$ since f' changes from negative to positive.

(b) Differentiate to find the critical values of $f(x) = x^3 - 3x^2 - 9x + 5$:

$$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3) \Longrightarrow x = -1, 3$$

Now construct a sign chart:



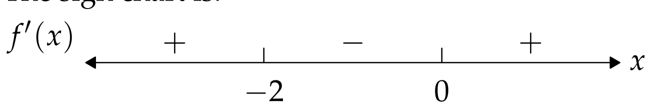
So we may conclude:

f is increasing on $(-\infty, -1)$ and $(3, \infty)$ since f' is positive on those intervals.
f is decreasing on $(-1, 3)$ since f' is negative on that interval.
f has a local maximum at $x = -1$ since f' changes from positive to negative.
f has a local minimum at $x = 3$ since f' changes from negative to positive.

(c) If $f(x) = x^2 e^x$, then

$$f'(x) = 2xe^x + x^2e^x = xe^x(2+x) = 0 \Longrightarrow x = -2, 0$$

The sign chart is:



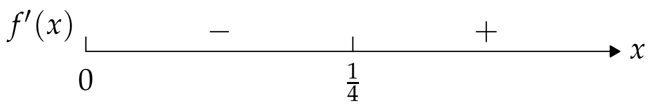
Therefore:

f is increasing on $(-\infty, -2)$ and $(0, \infty)$ since f' is positive on those intervals.
f is decreasing on $(-2, 0)$ since f' is negative on this interval.
f has a relative maximum at $x = -2$ since f' changes from positive to negative.
f has a relative minimum at $x = 0$ since f' changes from negative to positive.

(d) For $f(x) = x - \sqrt{x}$, we have

$$f'(x) = 1 - \frac{1}{2\sqrt{x}} \Longrightarrow x = \frac{1}{4}$$

Since f is undefined for $x < 0$, we need not consider any of those values. Here is the chart:



So we have the following conclusions:

f is increasing on $(\frac{1}{4}, \infty)$ since f' is positive on that interval.

f is decreasing on $(0, \frac{1}{4})$ since f' is negative on that interval.

f has a relative minimum at $x = \frac{1}{4}$ since f' changes from negative to positive.

5. (a) The vertex of the parabola is at the local minimum or local maximum of the quadratic, which occurs when $\mathrm{d}y/\mathrm{d}x = 0$:

$$y = ax^2 + bx + c \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b = 0 \Longrightarrow x = -\frac{b}{2a}$$

(b) We can repeat with the vertex form:

$$y = a(x-h)^2 + k \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2a(x-h) = 0 \Longrightarrow x = h$$

(c) By observing the expressions for $\mathrm{d}y/\mathrm{d}x$ in (a) and (b), we can see that the vertex is a local minimum when $a > 0$ and a local maximum when $a < 0$.

6. g having a critical point at $(-2, 3)$ means that the graph of the function passes through that point (i.e. $g(-2) = 3$) and that $g'(-2) = 0$, since g must have a critical point at $x = -2$. So:

$$g'(x) = 3x^2 + a \Longrightarrow f'(-2) = a + 12 = 0, \quad g(-2) = -8 - 2a + b = 3$$
$$\Longrightarrow \begin{cases} a = -12 \\ -2a + b = 11 \end{cases} \Longrightarrow -2(-12) + b = 11 \Longrightarrow b = -13$$

So $a = -12$ and $b = -13$.

7. (a) First, find the critical values for $f(x) = \sin 2x + 2\cos x$ over $[0, \frac{3\pi}{2}]$:[1]

$$f'(x) = 2\cos 2x - 2\sin x = 2(1 - 2\sin^2 x) - 2\sin x = 2 - 2\sin x - 4\sin^2 x = 0$$
$$\Longrightarrow 2\sin^2 x + \sin x - 1 = (2\sin x - 1)(\sin x + 1) = 0$$
$$\Longrightarrow 2\sin x - 1 = 0 \Longrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \qquad \sin x + 1 = 0 \Longrightarrow x = \frac{3\pi}{2}$$

Note that the above equation has infinitely many solutions, but we are only interested in the solutions over the interval $0 \leq x \leq \frac{3\pi}{2}$.

[1] Recall the cosine double angle formula: $\cos 2x = 1 - 2\sin^2 x$.

Insert the critical values and endpoints into a table of values:

x	0	$\pi/6$	$5\pi/6$	$3\pi/2$
$f(x)$	2	$3\sqrt{3}/2$	$-3\sqrt{3}/2$	0

Thus the absolute maximum of f on the interval $[0, \frac{3\pi}{2}]$ is $\frac{3\sqrt{3}}{2}$, and the absolute minimum is $-\frac{3\sqrt{3}}{2}$.

(b) Find the critical values of g over $[-3, 5]$:

$$g'(x) = 3x^2 - 12 = 3(x-2)(x+2) = 0 \Longrightarrow x = -2, 2$$

Now construct a table of values for g:

x	-3	-2	2	5
$g(x)$	7	14	-18	63

The maximum value of g on the given interval is 63, and the minimum value is -18.

(c) Find the critical values of $h(x) = x^{-1} \ln x$:

$$h'(x) = \frac{-\ln x + 1}{x^2} \Longrightarrow x = e$$

Make a table of values for $h(x)$:

x	$1/2$	e	5
$h(x)$	-1.386	0.368	0.322

The maximum of h on the given interval is 0.368, and the minimum value is -1.386.

8. $y = x - \ln x \Longrightarrow \mathrm{d}y/\mathrm{d}x = 1 - \frac{1}{x} = 0 \Longrightarrow x = 1$ is the only critical value. We can test

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{1}{2}} = -1 < 0, \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2} = \frac{1}{2} > 0$$

Therefore this singular critical point is a local minimum.

9. $x = -\frac{1}{\sqrt{2}}$ is not in the domain of f! The domain of f is $(0, \infty)$.

10. $g'(x) = -4e^{-x}\sin x = 4e^{-x}\cos x = 4e^{-x}(\cos x - \sin x) = 0 \Longrightarrow x = \frac{\pi}{4}$ is a critical value of g. Test values to determine whether $g(\frac{\pi}{4})$ is a relative minimum, relative maximum, or neither:

$$g'\left(\frac{\pi}{6}\right) > 0, \quad g'\left(\frac{\pi}{3}\right) < 0$$

Since $g'(x)$ changes from positive to negative at $x = \frac{\pi}{4}$, g has a relative maximum at $x = \frac{\pi}{4}$.

11. (a) The average rate of change of f from $x = 2$ to $x = 5$ is

$$\frac{f(5) - f(2)}{5 - 2} = \frac{2 - 5}{5 - 2} = -1$$

Since f is twice-differentiable, f is also continuous and differentiable. Therefore, by the Mean Value Theorem, there exists a value c between $x = 2$ and $x = 5$ such that $f'(c) = -1$.

(b) First, we find $g'(x)$:

$$\begin{aligned}
&g'(x) = f'(f(x)) \cdot f'(x) \\
\Longrightarrow &g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2) \\
\Longrightarrow &g'(5) = f'(f(5)) \cdot f'(5) = f'(5) \cdot f'(2)
\end{aligned}$$

Regardless of the values of $f'(2)$ and $f'(5)$, we have $g'(2) = g'(5)$. Using this, the average rate of change of g' from $x = 2$ to $x = 5$ is

$$\frac{g'(5) - g'(2)}{5 - 2} = 0$$

g' is a composition of differentiable functions, and is therefore continuous and differentiable. By the Mean Value Theorem (or Rolle's Theorem), there must be a value k for which $2 < k < 5$ such that $g''(k) = 0$.

(c) h is a difference of continuous functions, and is therefore continuous. We have

$$h(2) = f(2) - 2 = 5 - 2 = 3 > 0, \quad h(5) = f(5) - 5 = 2 - 5 = -3 < 0$$

By the Intermediate Value Theorem, there must exist a value r for $2 < r < 5$ such that $h(r) = 0$.

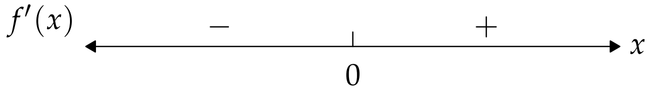
The challenge with this question is deciphering whether to use Mean Value Theorem or Intermediate Value Theorem. Re-read their statements to make sense of it!

Exercises 3.2 Solutions

1. (a) First, find all information pertaining to the first derivative. We begin by finding the critical values of f:

$$f'(x) = \frac{1}{x^2+3} \cdot 2x = 0 \Longrightarrow x = 0$$

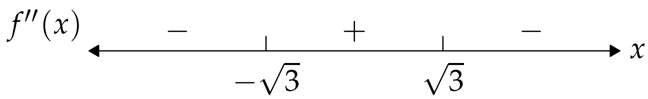
The domain of f is $(-\infty, \infty)$, so we may include all x-values in our sign chart.



We can also find the expression for f'' to find the critical values of f':

$$f''(x) = \frac{2(x^2+3) - 2x(2x)}{(x^2+3)^2} = \frac{-2x^2+6}{(x^2+3)^2} = 0 \Longrightarrow x = \pm\sqrt{3}$$

Our sign chart for f'' is shown below:



So we can conclude the following:

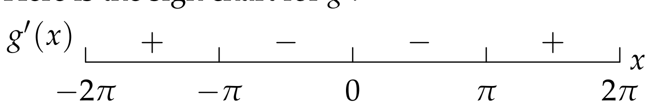
f is increasing on $(0, \infty)$ since f' is positive on that interval.
f is decreasing on $(-\infty, 0)$ since f' is negative on that interval.
f has a relative minimum at $x = 0$ since f' changes from negative to positive at $x = 0$.
f is concave up on $(-\sqrt{3}, \sqrt{3})$ since f'' is positive on that interval.
f is concave down on $(-\infty, \sqrt{3})$ and $(\sqrt{3}, \infty)$ since f'' is negative on those intervals.
f has points of inflection at the coordinates $(-\sqrt{3}, \ln 6)$ and $(\sqrt{3}, \ln 6)$ since f'' changes sign at those x-values.

(b) Start with finding the critical values of g:

$$g'(x) = -\frac{1}{2}\csc\frac{x}{2}\cot\frac{x}{2} = 0 \Longrightarrow \csc\frac{x}{2} = 0, \ \cot\frac{x}{2} = 0$$
$$\Longrightarrow \csc\frac{x}{2} = 0 \text{ has no solutions since the range of cosecant is } (-\infty, -1] \cup [1, \infty).$$
$$\Longrightarrow \cot\frac{x}{2} = 0 \Longrightarrow x = \pm\pi$$

The above equation has infinitely many solutions, but we only need the solutions such that $-2\pi \le x \le 2\pi$. Since $g'(0)$ is undefined, $x = 0$ is also a critical value of g.

Here is the sign chart for g':

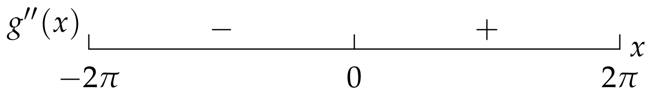


Now we find the critical values for g':

$$g''(x) = -\frac{1}{2}\left(-\frac{1}{2}\csc\frac{x}{2}\cot\frac{x}{2}\cdot\cot\frac{x}{2} + \csc\frac{x}{2}\cdot -\frac{1}{2}\csc^2\frac{x}{2}\right)$$
$$= \frac{1}{4}\csc\frac{x}{2}\left(\cot^2\frac{x}{2} + \csc^2\frac{x}{2}\right) = \frac{1}{4}\csc\frac{x}{2}\left(\csc^2\frac{x}{2} - 1 + \csc^2\frac{x}{2}\right)$$
$$= \frac{1}{4}\csc\frac{x}{2}\left(2\csc^2\frac{x}{2} - 1\right) = 0 \Longrightarrow \csc\frac{x}{2} = 0,\ 2\csc^2\frac{x}{2} - 1 = 0$$

Neither factor has any zeros. Therefore the only critical value of g' on $-2\pi \le x \le 2\pi$ is $x = 0$, where g'' is undefined.

Here is the sign chart for g'':



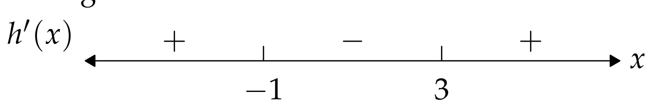
Here are the conclusions on the interval $[-2\pi, 2\pi]$:

g is increasing on $(-2\pi, -\pi)$ and $(\pi, 2\pi)$ since $g'(x) > 0$ on those intervals.
g is decreasing on $(-\pi, 0)$ and $(0, \pi)$ since $g'(x) < 0$ on those intervals.
g has a local maximum at $x = -\pi$ since g' changes from positive to negative at $x = -\pi$.
g has a local minimum at $x = \pi$ since g' changes from negative to positive at $x = \pi$.
g is concave up on $(0, 2\pi)$ since $g''(x) > 0$ on that interval.
g is concave down on $(-2\pi, 0)$ since $g''(x) < 0$ on that interval.
Although g'' changes sign at $x = 0$, g does *not* have a point of inflection at $x = 0$ since $g(0)$ is undefined!

(c) Find the critical values of h:

$$h'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3) = 0 \Longrightarrow x = -1, 3$$

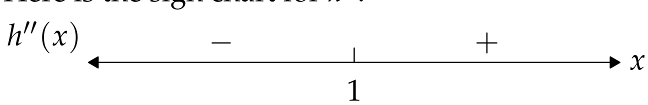
The sign chart for h' is shown below:



We find the critical values for h' by finding h'':

$$h''(x) = 6x - 6 = 0 \Longrightarrow x = 1$$

Here is the sign chart for h'':



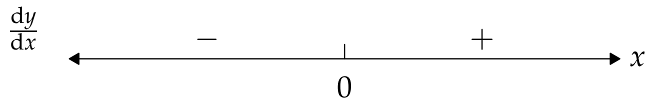
Here are the takeaways based on the above:

h is increasing on $(-\infty, -1)$ and $(3, \infty)$ since $h'(x) > 0$ on those intervals.
h is decreasing on $(-1, 3)$ since $h'(x) < 0$ on that interval.
h has a relative maximum at $x = -1$ since h' changes from positive to negative at $x = -1$.
h has a relative minimum at $x = 3$ since h' changes from negative to positive at $x = 3$.
h is concave up on $(1, \infty)$ since $h''(x) > 0$ on that interval.
h is concave down on $(-\infty, 1)$ since $h''(x) < 0$ on that interval.
h has a point of inflection at $x = 1$ since h'' changes sign at $x = 1$.

(d) Find the critical values by finding the zeros of $\mathrm{d}y/\mathrm{d}x$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(e^x - e^{-x}) = 0 \Longrightarrow x = 0$$

The sign chart for $\mathrm{d}y/\mathrm{d}x$ is shown below:



Now we find the second derivative:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2}(e^x + e^{-x}) = 0$$

The second derivative has no zeros or undefined values, so the graph of the function has no points of inflection. To test for the concavity of y for $(-\infty, \infty)$, simply evaluate the second derivative at any x-value, say $x = 0$:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = \frac{1}{2}(e^0 + e^0) = 1 > 0$$

Thus, we have:

y is increasing for $(0, \infty)$ since $\mathrm{d}y/\mathrm{d}x > 0$ on that interval.
y is decreasing for $(-\infty, 0)$ since $\mathrm{d}y/\mathrm{d}x < 0$ on that interval.
y has a local minimum at $x = 0$ since $\mathrm{d}y/\mathrm{d}x$ changes from negative to positive at $x = 0$.
y is concave up on $(-\infty, \infty)$ since $\mathrm{d}^2y/\mathrm{d}x^2$ is positive for all x.

2. (a) For the standard normal distribution function, we have

$$f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot -x = 0 \Longrightarrow x = 0$$

The sign chart for the function is below:

Therefore f is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$ and has a local maximum at the point $(0, \frac{1}{\sqrt{2\pi}})$.

(b) The points of inflection occur at the zeros of f'':

$$f''(x) = -\frac{1}{\sqrt{2\pi}}(1 \cdot e^{-\frac{1}{2}x^2} + x \cdot e^{-\frac{1}{2}x^2} \cdot -x) = -\frac{e^{-\frac{1}{2}x^2}(1-x^2)}{\sqrt{2\pi}} = 0$$
$$\implies 1 - x^2 = 0 \implies x = \pm 1$$

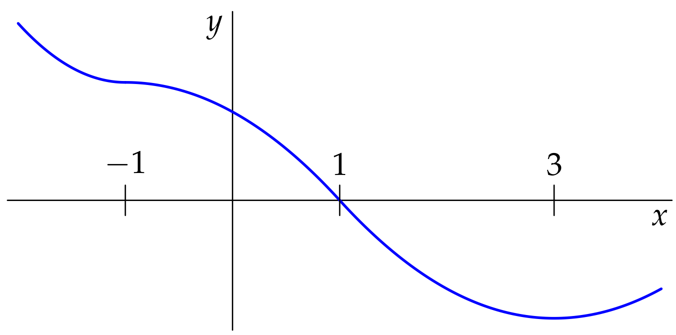
Make a sign chart for f'' to verify that f'' changes sign at $x = -1$ and $x = 1$, and thus the points of inflection for f occur at those x-values.

(c) As $x \to -\infty$, $e^{-\frac{1}{2}x^2} \to 0$. Similarly, as $x \to \infty$, $e^{-\frac{1}{2}x^2} \to 0$. The horizontal asymptote of the function is $y = 0$.

3. (a) f has a critical value at $x = -1$, but it is neither a local maximum nor local minimum, since f' does not change sign there. f has a local minimum at $x = 3$, since f' changes from negative to positive.

(b) f has a point of inflection at $x = -1$ and $x = 1$ since f' has a local minimum or local maximum at those x-values.

(c) Here is a possible graph of f:



Here is the important features that must be shown:

f is decreasing on $(-\infty, -1)$ and $(-1, 3)$.
f is increasing on $(3, \infty)$.
f is concave up on $(-\infty, -1)$ and $(1, \infty)$.
f is concave down on $(-1, 1)$.
f has a horizontal tangent at $x = -1$ and $x = 3$: the point at $x = 3$ is a local minimum.
f has points of inflection at $x = -1$ and $x = 1$.

The vertical translation is unimportant.

(d) No: both $f'(x) = 0$ and $f''(x) = 0$.

4. We have

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \implies f''(x) = 0$$

$f'(x) = 0$ tells us that f is constant![2] Therefore f also has no concavity.

[2] This is related to the trigonometric *cofunction identities*.

5. $g(3)$ is unimportant for us here. Since $g'(3) = 0$ and $g''(3) < 0$, we can conclude that g has a relative maximum at $x = 3$.

6. (a) Given the standard form $y = ax^3 + bx^2 + cx + d$, we can differentiate to find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx + c$$

The derivative of the cubic is a quadratic, which has at most two zeros and no undefined values. Therefore, there are at most two critical values, and thus at most two local extrema.

We can also analyze the second derivative:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6ax + 2b$$

The second derivative is a linear, non-constant function, which *must* cross the x-axis exactly once. Hence a cubic function must have exactly one point of inflection.

(b) The critical values of the cubic occur at the zeros of $\mathrm{d}y/\mathrm{d}x$, which we can solve for using the quadratic formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx + c = 0 \implies x = \frac{-2b \pm \sqrt{(2b)^2 - 4(3a)(c)}}{2(3a)} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

The number of local extrema depends on the *discriminant* of the quadratic:

If $b^2 - 3ac > 0$, the graph of $\mathrm{d}y/\mathrm{d}x$ crosses the x-axis twice, and therefore has two local extrema.

If $b^2 - 3ac = 0$, $\mathrm{d}y/\mathrm{d}x = 0$ has one *repeated* solution. If this is the case, $\mathrm{d}y/\mathrm{d}x$ does not change sign at any x-value. In this case, the cubic polynomial has no local relative extrema.

If $b^2 - 3ac < 0$, the derivative has no zeros and thus y has no critical values. In this case, the cubic also has no local extrema.

To conclude, if $b^2 - 3ac > 0$, the cubic has two local extrema. If $b^2 - 3ac \leq 0$, the cubic has no local extrema. Under no circumstance does the cubic polynomial have only one relative extremum. If you are having difficulty visualizing this, recall quadratic functions...

The singular point of inflection of the cubic occurs at the zero of the second derivative:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6ax + 2b = 0 \implies x = -\frac{b}{3a}$$

7. (a) h has a relative minimum at $x = -1$ since $h'(x)$ changes from negative to positive at $x = -1$.

   (b) The graph of h has points of inflection at $x = -2$, $x = 0$, and $x = 1$ since h' has a local minimum or local maximum at those x-values.

   (c) We find the x-values for which $H'(x) < 0$:

$$H(x) = x - h(x) \Longrightarrow H'(x) = 1 - h'(x) < 0 \Longrightarrow h'(x) > 1$$

   So we are searching for the intervals for which the graph of h' is above the line $y = 1$: the intervals for which H' is decreasing are $(-4, -3.5)$ and $(2, 4)$.

   (d) $h''(-3)$ is the slope of the graph of h' at $x = -3$. The slope of the line segment including $x = -3$ is -2, so $h''(-3) = -2$.

   At $x = 1$, the graph of h' has a *cusp*. Thus h' is not differentiable at $x = 1$, and therefore $h''(1)$ is undefined.

Exercises 3.3 Solutions

1. (a) The particle is moving to the right on the time intervals $(1, 4)$ and $(4, 6)$ since $v(t) > 0$ on those intervals.

   (b) The particle reverses direction at two t-values (at $t = 1$ and $t = 6$), since $v(t)$ changes sign at those times.

   (c) On the time interval $2 < t < 4$, the speed of the particle is decreasing since $v(t) > 0$ and $a(t) = v'(t) < 0$ on that interval. That is, the velocity and acceleration have different signs on this interval.

   (d) The acceleration of the particle is positive on the time intervals $(0, 2)$ and $(4, 5)$ since $v(t)$ is increasing on those intervals.

2. (a) The velocity function is $v(t) = s'(t) = 321.5 - 32.2t$, and the acceleration function is $a(t) = v'(t) = -32.2$.

   (b) The maximum height of the stone occurs at the zero of $v(t)$, which is also the t-value of the vertex of the parabola given by $s(t)$:

   $$v(t) = 321.5 - 32.2t = 0 \Longrightarrow t = 9.984$$

   The maximum height itself is $s(9.984) = 1605.004$ feet.

   (c) The stone hits the ground when $s(t) = 0 \Longrightarrow t = 19.969$ seconds.

   (d) $s'(3) = 224.9$. At $t = 3$ seconds, the height of the stone is increasing at a rate of 224.9 feet per second.

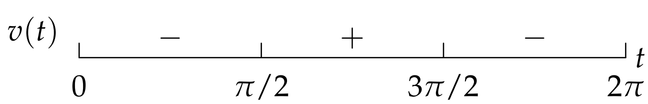
3. (a) The initial position of the particle is $x(0) = 1 - 2\sin 0 = 1$ centimeters.

   Since $v(t) = -2\cos t$, the initial velocity of the particle is $v(0) = -2\cos 0 = -2$ centimeters per minute.

   Since $a(t) = 2\sin t$, the initial acceleration of the particle is $a(0) = 2\sin 0 = 0$ centimeters per minute per minute.

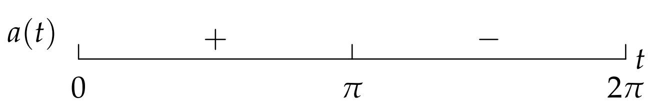
   (b) The particle's speed is decreasing whenever the sign of $v(t)$ and $a(t)$ are opposite. It is useful to make a sign chart for $v(t)$, so find the zeros of v:

   $$v(t) = -2\cos t = 0 \Longrightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$$



   The same can be done for $a(t)$:

   $$a(t) = 2\sin t = 0 \Longrightarrow t = \pi$$



   Note that we are only considering values on the interval $0 \le t \le 2\pi$. From the sign charts, we can see that the signs of $v(t)$ and $a(t)$ are opposite on the intervals $(0, \frac{\pi}{2})$ and $(\pi, \frac{3\pi}{2})$. The speed of the particle is decreasing on these intervals.

(c) We found the critical values of $v(t)$ to be $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ from part (b). We can make a table of values for $x(t)$:

t	0	$\pi/2$	$3\pi/2$	2π
$x(t)$	1	-1	3	1

It follows that at $t = \frac{3\pi}{2}$, the particle is furthest to the right.

(d) $x''(\frac{\pi}{4}) = a(\frac{\pi}{4}) = 2\sin\frac{\pi}{4} = \sqrt{2}$. At time $t = \frac{\pi}{4}$ minutes, the particle is accelerating (i.e. the velocity of the particle is increasing) at a rate of $\sqrt{2}$ centimeters per minute per minute.

4. (a) The average rate of change of V from $t = 0$ to $t = 5$ is given by

$$\frac{V(5) - V(0)}{5 - 0} = \frac{112500 - 162000}{5} = -9900 \text{ m}^3/\text{h}$$

(b) The instantaneous rate of change of V requires V':

$$V'(t) = -360(30 - t) \Longrightarrow V'(5) = -360(30 - 5) = -9000 \text{ m}^3/\text{h}$$

(c) The reservoir is empty when $V(t) = 180(30 - t)^2 = 0 \Longrightarrow t = 30$ hours. That is, at 7 p.m. the following day.

5. (a) As the hint suggests, we may estimate $v(2)$ using the average velocity of the particle from $t = 1$ to $t = 3$ with

$$v'(3) \approx \frac{s(3) - s(1)}{3 - 1} = \frac{-1 - 4}{2} = -2.5 \text{ inches per second.}$$

(b) Since s is twice-differentiable, it is also continuous and differentiable. The average velocity from $t = 0$ to $t = 5$ is

$$\frac{s(5) - s(0)}{5 - 0} = \frac{4 - 4}{5} = 0$$

By the Mean Value Theorem, there must be a time t on the interval $0 < t < 5$ such that $v(t) = 0$, i.e. the particle is at rest.

(c) Since s is twice-differentiable, it is also continuous. Notice that $s(1) > 0$ and $s(3) < 0$, so by the Intermediate Value Theorem, there must be a time between $t = 1$ and $t = 3$ such that $s(t) = 0$, i.e. the particle is at the origin. A similar argument can be used to show that there is a time on each of the intervals $3 < t < 5$ and $9 < t < 11$ for which $s(t) = 0$. Thus the minimum number of times the particle must be at the origin is three.

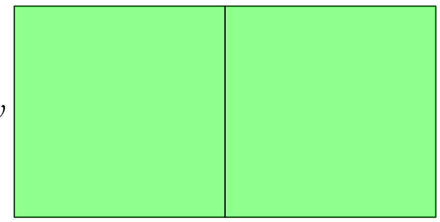
6. (a) The profit is simply the difference between the revenue and cost:

$$P(x) = R(x) - C(x) = 170 \ln\left(1 + \frac{x}{90}\right) - (x - 70)^2 + 550$$

(b) Using a calculator, $P'(50) = 41.214$. When the manufacturer produces and sells 50 glasses per day, the company's profit is increasing at a rate of 41.21 dollars per glasses per day.

Exercises 3.4 Solutions

1. The picture of the described situation is on the right. Given the diagram, the total amount of fencing is the sum of three widths and two lengths, totaling to 200 feet. Our goal is to maximize area:



$$200 = 2\ell + 3w \Longrightarrow \ell = 100 - \frac{3}{2}w \quad \text{and}$$

$$\text{Area: } A = \ell w = (100 - \frac{3}{2}w)w = 100w - \frac{3}{2}w^2$$

Now differentiate to find the critical value of A:

$$A'(w) = 100 - 3w = 0 \Longrightarrow w = \frac{100}{3}$$

Either substitute the above value of w to find the corresponding value of ℓ, or evaluate

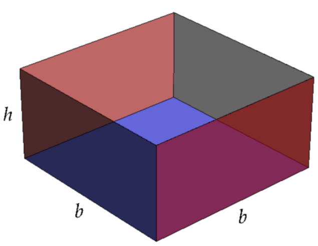
$$A\left(\frac{100}{3}\right) = \frac{5000}{3} \text{ square feet.}$$

2. No need to draw a picture here; if the two numbers are x and y, we have the equations

$$x - y = 40 \Longrightarrow x = y + 40, \quad P = xy = (y+40)y = y^2 + 40y$$
$$P'(y) = 2y + 40 \Longrightarrow y = -20 \Longrightarrow x = -20 + 40 = 20$$

The two numbers are thus $x = 20$ and $y = -20$.

3. We have the same box with square base, no lid, and volume 4 ft^3, but our goal is different now: we are to minimize the cost of the box. We can achieve this by multiplying the areas of the faces by their respective costs:



$$V = 4 = b^2h \Longrightarrow h = \frac{4}{b^2} \quad \text{and}$$

$$\text{Cost: } C = 2 \cdot b^2 + 1 \cdot 4bh = 2b^2 + 4b\left(\frac{4}{b^2}\right)$$
$$= 2b^2 + 16b^{-1}$$

Now differentiate to find the critical values of C:

$$\frac{dC}{db} = 4b - 16b^{-2} = \frac{4b^3 - 16}{b^2} = 0 \Longrightarrow b = 0, 4^{1/3}$$

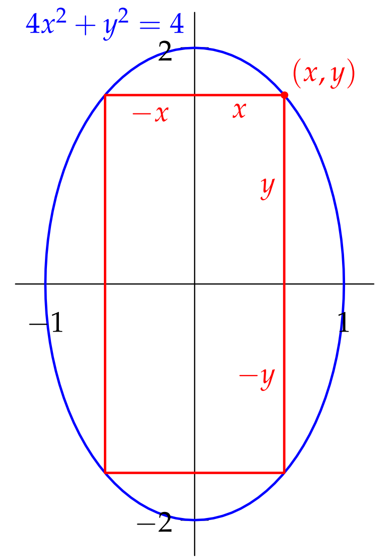
The value $b = 0$ is physically irrelevant; if $b = 4^{1/3}$, then $h = 4 \cdot 2^{-2/3}$. These are the dimensions of the cheapest box. For the minimum cost itself, $C(4^{1/3}) = 2(4^{1/3}) + 16(4^{1/3})^{-1} = 2^{7/3} + 2^{10/3}$ dollars. Not a pretty number at all!

4. (a) The initial population of microbes is $P(0) = 50$ million microbes.
   (b) Find the critical value of P to get the time for which population is a maximum:

$$P'(t) = \frac{600 \cdot (45 + t^2) - 600t \cdot 2t}{(45 + t^2)^2} = \frac{27000 - 600t^2}{(45 + t^2)^2} = 0$$
$$\implies 27000 - 600t^2 = 600(45 - t^2) = 0 \implies t = \pm 3\sqrt{5}$$

   The negative time value is irrelevant, so we conclude that the maximum population occurs at $t = 3\sqrt{5}$ days. You may perform a derivative test to verify that this is indeed a local maximum.
   (c) $P(3\sqrt{5}) = 94.721$ million microbes.

5. Our goal is to maximize the area of the inscribed rectangle, using the equation of the ellipse as our constraint:



$$4x^2 + y^2 = 4 \implies y = \sqrt{4 - 4x^2}$$
$$A = 4xy = 4x\sqrt{4 - 4x^2}$$

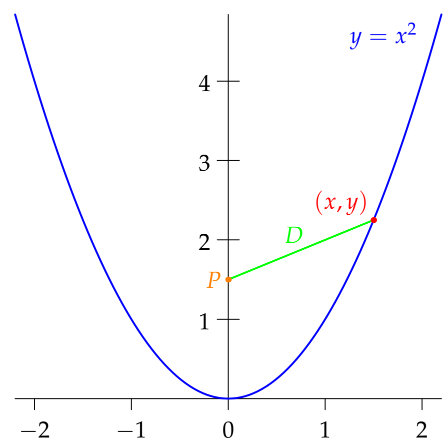
Now differentiate A with respect to x:

$$A'(x) = 4\sqrt{4 - 4x^2} + 4x \cdot \frac{-4x}{\sqrt{4 - 4x^2}}$$
$$= \frac{16 - 32x^2}{\sqrt{4 - 4x^2}} = 0 \implies x = \pm\frac{1}{\sqrt{2}}$$

Therefore the maximum possible area is

$$A\left(\frac{1}{\sqrt{2}}\right) = 4 \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{4 - 2} = 4$$

6. We are minimizing the distance D between a point on the parabola and P, using $y = x^2$ as the constraint. We will minimize the square of distance to not worry about a nasty derivative:



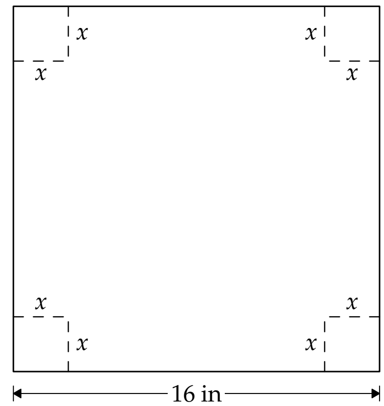
$$y = x^2$$
$$d = (x - 0)^2 + \left(y - \frac{3}{2}\right)^2 = x^2 + \left(x^2 - \frac{3}{2}\right)^2$$
$$= x^2 + x^4 - 3x^2 + \frac{9}{4} = x^4 - 2x^2 + \frac{9}{4}$$

Differentiate d to get the critical values:

$$d'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1) \implies x = 0, \pm 1$$

How do we know which values correspond to maxima and which to minima? You may make a sign chart or use the second derivative test to see that $x = \pm 1$ gives a minimum distance, and therefore the closest points on the curve are $(\pm 1, 1)$.

7. This problem is difficult to solve without visualizing it: make sure you do draw a picture. Our goal is to maximize the volume of the box created by cutting a square of length x from each corner. The volume is only dependent on x, so no constraint is necessary:



$$V = \underbrace{(16-2x)^2}_{\text{base}} \cdot \underbrace{(x)}_{\text{height}} = 4x^3 - 64x^2 + 256x$$

Then differentiate to find the critical values:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 128x + 256 = 4(3x-8)(x-8) = 0$$

So the critical values are $x = \frac{8}{3}$ and $x = 8$. Clearly $x = 8$ makes no physical sense, so we should trim $\frac{8}{3}$ inches from each corner to produce a box with maximal volume.

8. First find the distance: the object hits the ground when

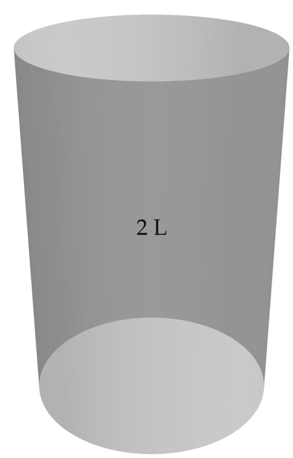
$$y = 0 \Longrightarrow x = 0,\ \frac{v^2}{g}\sin 2\theta$$

Therefore, it is actually the distance $R = \frac{v^2}{g}\sin 2\theta$ that we want to maximize. Its critical points occur when

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \frac{2v^2}{g}\cos 2\theta = 0 \Longrightarrow 2\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots \Longrightarrow \theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \ldots$$

Only angles $0 \le \theta \le \frac{\pi}{2}$ make physical sense (look at the picture!). Thus $\theta = \frac{\pi}{4} = 45°$ gives the maximum distance.

9. Given a volume constraint of 2000 cubic centimeters, using the least possible amount of materials means minimizing surface area. Use the formulas in the hint to get



$$2000 = \pi r^2 h \Longrightarrow h = \frac{2000}{\pi r^2}$$

$$A = 2\pi r h + 2\pi r^2 = 2\pi r\left(\frac{2000}{\pi r^2}\right) + 2\pi r^2$$

$$= 4000r^{-1} + 2\pi r^2$$

Then differentiate surface area to find its critical values:

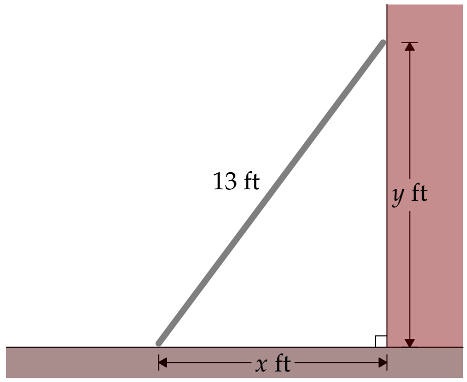
$$\Longrightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = -4000r^{-2} + 4\pi r = 0 \Longrightarrow r = 6.828$$

The required radius is $r = 6.828$ cm. The corresponding height is

$$h = \frac{2000}{\pi(6.828)^2} = 13.656\,\text{cm}$$

Exercises 3.5 Solutions

1. The length of the ladder remains constant, while the distance x between the bottom of the ladder and the wall, and the distance y between the top of the ladder and the ground are changing over time. We therefore have, using the Pythagorean Theorem:



$$x^2 + y^2 = 169$$
$$\implies 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

In this instant, the top of the ladder is a distance of $y = 5$ ft from the ground, the bottom of the ladder (once again using the Pythagorean Theorem) is $x = 12$ ft from the wall, and the top of the ladder is moving at a rate of $dy/dt = -8$ ft/s. Our target is dx/dt:

$$2(12)\frac{dx}{dt} + 2(5)(-8) = 0 \implies \frac{dx}{dt} = \frac{10}{3} \text{ ft/s}$$

Thus the bottom of the ladder is sliding away from the wall at a rate of $10/3$ feet per second at this instant.

2. We don't need a picture for this question, but we are given the equation

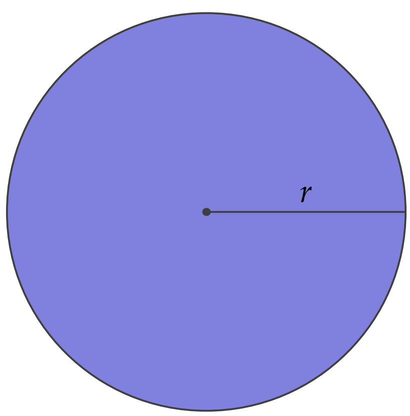
$$ab^2 = 100 \implies \frac{da}{dt} \cdot b^2 + a \cdot 2b \cdot \frac{db}{dt} = 0$$

We know that $a = 6$ means $6b^2 = 100 \implies b = \frac{5\sqrt{3}}{3}$, and $db/dt = 2$, so

$$\frac{da}{dt} \cdot \frac{50}{3} + 6 \cdot 2\left(\frac{5\sqrt{3}}{3}\right) \cdot 2 = 0 \implies \frac{da}{dt} = -\frac{12\sqrt{3}}{5} \text{ units per second}$$

So a is decreasing at a rate of $-\frac{12\sqrt{3}}{5}$ units per second at this time.

3. In this problem, no quantity is constant. We are asked about the rate of change of area, so use



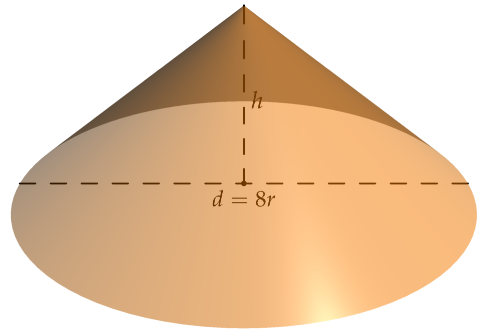
$$A = \pi r^2 \implies \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

Our required variable is dA/dt, so use the information $r = 50$ and $dr/dt = 3$:

$$\frac{dA}{dt} = 2\pi(50) \cdot 3 = 300\pi \text{ cm/s}$$

Therefore the area of the spilled water is increasing at a rate of 300π centimeters per second in this moment.

4. Firstly, since the diameter of the pile is always 8 times the height, we have $d = 8h \implies r = 4h$. We are also given information about the rate of change of volume, so use the volume formula for a cone:



$$V = \frac{1}{3}\pi r^2 h \implies \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{3}\left(2rh \cdot \frac{\mathrm{d}r}{\mathrm{d}t} + r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right)$$

We are given $\mathrm{d}V/\mathrm{d}t = 50$, $d = 24 \implies r = 12$, and consequently $h = 3$. Although we are not given $\mathrm{d}r/\mathrm{d}t$, we can use

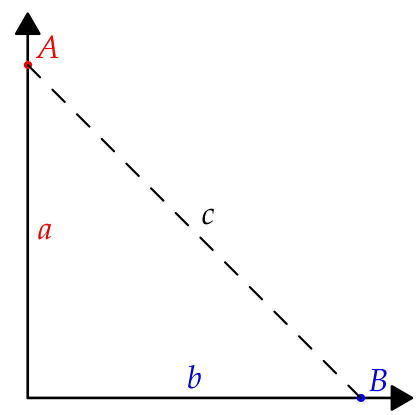
$$r = 4h \implies \frac{\mathrm{d}r}{\mathrm{d}t} = 4 \cdot \frac{\mathrm{d}h}{\mathrm{d}t} \implies \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{3}\left(2rh\left(4 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right) + r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right)$$

Now substitute all given information to solve for our required rate of change $\mathrm{d}h/\mathrm{d}t$:

$$50 = \frac{\pi}{3}\left(2(12)(3)\left(4 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right) + (12)^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right) \implies \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{25}{72\pi}\,\text{in/s}$$

Therefore the height of the sand pile at this time is increasing at a rate of $\frac{25}{72\pi}$ inches per second.

5. In the picture, notice that Adam's position, Bob's position, and the starting position at any time form a right triangle. Adam's distance a from the starting position, Bob's distance b from the starting position, and the distance between the two runners c are all non-constant. Use the Pythagorean Theorem:



$$a^2 + b^2 = c^2 \implies 2a \cdot \frac{\mathrm{d}a}{\mathrm{d}t} + 2b \cdot \frac{\mathrm{d}b}{\mathrm{d}t} = 2c \cdot \frac{\mathrm{d}c}{\mathrm{d}t}$$

In this moment, Adam is running at $\mathrm{d}a/\mathrm{d}t = 8\,\text{mi/h}$, Bob is running at $\mathrm{d}b/\mathrm{d}t = 6\,\text{mi/h}$, Bob is $b = 7\,\text{mi}$ from the start, and the two protagonists are $c = 16\,\text{mi}$ apart. We can find $a = \sqrt{207} = 3\sqrt{23}$ by using the Pythagorean Theorem. Note that our goal is to find the value of $\mathrm{d}c/\mathrm{d}t$:

$$2(3\sqrt{23})(8) + 2(7)(6) = 2(16) \cdot \frac{\mathrm{d}c}{\mathrm{d}t} \implies \frac{\mathrm{d}c}{\mathrm{d}t} = \frac{12\sqrt{23} + 21}{8}\,\text{mi/h}$$

So the distance between Adam and Bob is increasing at a rate of $\frac{12\sqrt{23}+21}{8}$ miles per hour at this moment.

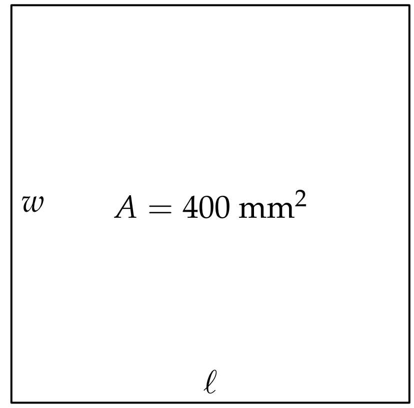
6. Area is constant, so we can write the equation

$$\ell w = 400 \Longrightarrow \frac{d\ell}{dt} \cdot w + \ell \cdot \frac{dw}{dt} = 0$$

We are interested in the rate of change of w when the rectangle is a square, i.e. when $\ell = w$. In this moment, the original equation tells us that $w = \ell = 20$ mm. We are also told that $d\ell/dt = -2$ mm/min, so

$$(-2)(20) + (20)\frac{dw}{dt} = 0 \Longrightarrow \frac{dw}{dt} = 2 \text{ mm/min}$$

The width of the rectangle is increasing at a rate of 2 millimeters per minute at this time.

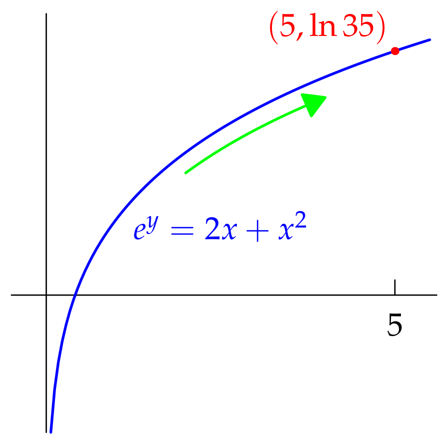


7. We are given directly the equation of the curve, which we can differentiate with respect to t:

$$e^y = 2x + x^2 \Longrightarrow e^y \cdot \frac{dy}{dt} = 2 \cdot \frac{dx}{dt} + 2x \cdot \frac{dx}{dt}$$

Given $x = 5$ at this time and $dx/dt = 3$, our goal is to find dy/dt. We can find the corresponding y-coordinate by substituting in $x = 5$:

$$e^y = 2(5) + 5^2 \Longrightarrow y = \ln 35$$



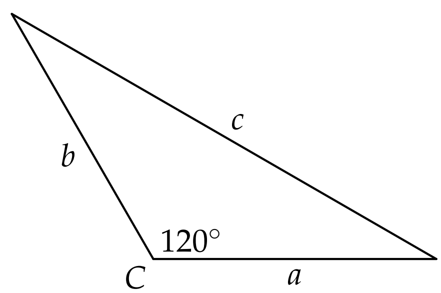
Now substitute our values into the differential equation:

$$e^{\ln 35} \cdot \frac{dy}{dt} = 2(3) + 2(5)(3) \Longrightarrow \frac{dy}{dt} = \frac{36}{35} \text{ units per second}$$

8. The angle between the two planes remains constant at 120°, but each of the distances are changing with time:

$$c^2 = a^2 + b^2 + 2ab\cos 120^\circ = a^2 + b^2 + ab$$

$$\Longrightarrow 2c \cdot \frac{dc}{dt} = 2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt} + \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt}$$



After 2 hours, A has traveled $a = 2(880) = 1760$ km, while plane B has traveled $b = 2(910) = 1820$ km. The rates of change of a and b are given, and the current distance c between the planes can be found using the law of cosines:

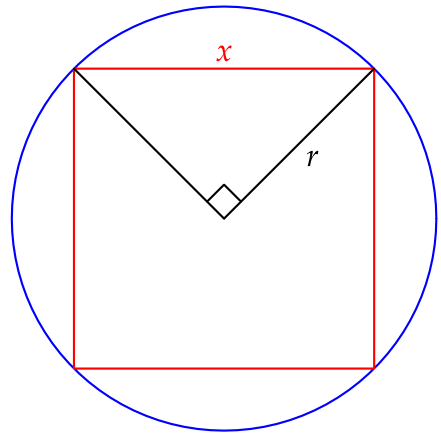
$$c^2 = 1760^2 + 1820^2 + (1760)(1820) \Longrightarrow c = 3100.516$$

Now we may substitute to find our target dc/dt:

$$2(3100.516) \cdot \frac{dc}{dt} = 2(1760)(880) + 2(1820)(910) + (880)(1820) + (1760)(910)$$

$$\Longrightarrow \frac{dc}{dt} = 1550.258 \text{ km/h}$$

9. If we label each side of the square x, then



$$A = x^2 \implies \frac{dA}{dt} = 2x \cdot \frac{dx}{dt}$$

In order to find our target dA/dx, we can substitute $x = 5$, but we still need dx/dt. For that, note the relationship between r and x; from the picture, we see

$$r = \sqrt{2}x \implies \frac{dr}{dt} = \sqrt{2} \cdot \frac{dx}{dt}$$

So, in this moment, we have

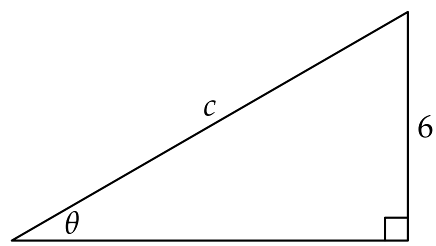
$$\frac{dA}{dt} = 2(5) \cdot \frac{-8}{\sqrt{2}} = -\frac{80}{\sqrt{2}} \text{ units/s}$$

10. No picture is required; we have the relationship $PV = T$, so

$$\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = \frac{dT}{dt} \implies \frac{dP}{dt}(2) + (10000)(-0.01) = 3 \implies \frac{dP}{dt} = 51.5 \text{ Pa/s}$$

Note that dV/dt is negative since volume is decreasing. The gas being 'heated' implies temperature is increasing.

11. The leg opposite our angle of interest θ has constant length 6, while the hypotenuse c is variable. We write an equation which relates our important quantities:



$$\sin\theta = \frac{6}{c} \implies \cos\theta \cdot \frac{d\theta}{dt} = -\frac{6}{c^2} \cdot \frac{dc}{dt}$$

In this instant, we know $c = 6$, but we are missing θ. To find it, we recall that

$$\sin\theta = \frac{6}{12} = \frac{1}{2} \implies \theta = \frac{\pi}{6}$$

Now we may substitute all relevant information to find $d\theta/dt$:

$$\cos\frac{\pi}{6} \cdot \frac{d\theta}{dt} = -\frac{6}{12^2} \cdot (-1.5) \implies \frac{d\theta}{dt} = \frac{1}{8\sqrt{3}} \text{ radians/s}$$

Note that initially using the equation $\csc\theta = \frac{c}{6}$ would have also yielded the correct answer.

Exercises 3.6 Solutions

1. We have $f(16) = \sqrt[4]{16} = 2$, and

$$f'(x) = \frac{1}{4}x^{-3/4} \Longrightarrow f'(16) = \frac{1}{32}$$

Therefore the equation of the tangent line and desired approximation are

$$L(x) = f'(16)(x-16) + f(16) = \frac{1}{32}(x-16) + 2 \Longrightarrow f(15) \approx L(15) = \frac{1}{32}(15-16) + 2 = \frac{63}{32}$$

To determine whether this is an underestimate or overestimate, find

$$f''(x) = -\frac{3}{16}x^{-7/4}$$

For all x on its domain, $f''(x) < 0$ and thus the graph of f is concave down for all x on its domain. Therefore this approximation is an overestimate.

2. Since 1.5 is quite close to $\frac{\pi}{2}$, we use the local linearization of $\cos x$ at $x = \frac{\pi}{2}$ to estimate $g(1.5)$. First, we have $g(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$ and

$$g'(x) = -\sin x \Longrightarrow g'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

So the tangent line and required estimate are

$$L(x) = -1(x - 3.14) + 0 \Longrightarrow \cos 1.5 \approx L(1.5) = -\left(1.5 - \frac{3.14}{2}\right) = 0.07$$

Asking a calculator reveals that $\cos 1.5 = 0.0707$ to four decimals, which is close to our estimate.

3. (a) First, $f(1) = \frac{1}{2}$. To find the slope of the tangent line, we need $f'(1)$:

$$f'(x) = -(3-x^2)^{-2} \cdot -2x = \frac{2x}{(3-x^2)^2} \Longrightarrow f'(1) = \frac{2(1)}{(3-1^2)^2} = \frac{1}{2}$$

So the equation of the tangent line to f at $x = 1$ is $L(x) = \frac{1}{2}(x-1) + \frac{1}{2}$

(b) The tangent line approximation for $f(1.1)$ is

$$f(1.1) \approx L(1.1) = \frac{1}{2}(1.1-1) + \frac{1}{2} = \frac{11}{20} = 0.55$$

To find out whether this approximation less than or greater than the real value of $f(1.1)$, find an expression for f'':

$$f''(x) = \frac{2\cdot(3-x^2)^2 - 2x\cdot 2(3-x^2)\cdot -2x}{(3-x^2)^4} = \frac{6(x^2+1)}{(3-x^2)^3}$$

Between $x = 1$ and $x = 1.1$, $f''(x) > 0$, so the approximation in (a) is an underestimate.

4. (a) We have $h(0) = e^0 = 1$, and $h'(x) = e^x \Longrightarrow h'(0) = 1$. So the tangent line to h at $x = 0$ is

$$L(x) = 1(x-0) + 1 = x + 1$$

(b) The approximation for $e^{0.2}$ using the above tangent line is

$$h(0.2) = e^{0.2} \approx L(0.2) = 0.2 + 1 = 1.2$$

(c) $h''(x) = e^x$, which is positive for all x on $(-\infty, \infty)$. Therefore the graph of h is concave up for all x, and hence every tangent line approximation of h produces an underestimate.

5. Let $f(x) = \sqrt{x}$. We are finding the value b such that $L_{16}(x) - f(b) = L_{25}(x) - f(b)$. That is, the x-value for which the linear approximations at $x = 16$ and $x = 25$ for $f(x) = \sqrt{x}$ are equivalent. First, we find the tangent lines themselves.

We have $f(16) = \sqrt{16} = 4$ and $f(25) = \sqrt{25} = 5$. We also need the slopes:

$$f'(x) = \frac{1}{2\sqrt{x}} \Longrightarrow f'(16) = \frac{1}{8}, \quad f'(25) = \frac{1}{10}$$

The tangent lines are $L_{16}(x) = \frac{1}{8}(x-16) + 4$ and $L_{25}(x) = \frac{1}{10}(x-25) + 5$. Now solve:

$$L_{16}(b) - f(b) = L_{25}(b) - f(b) \Longrightarrow L_{16}(b) = L_{25}(b) \Longrightarrow \frac{1}{8}(b-16) + 4 = \frac{1}{10}(b-25) + 5$$

Solving this equation gives $b = 20$.

6. (a) First, $f(4) = 52$ and $f'(x) = 3x^2 - 3 \Longrightarrow f'(4) = 45$. The tangent line equation is

$$L(x) = 45(x-4) + 52$$

(b) Using the tangent line approximation and comparing with the actual values:
$L(5) = 97$, $f(5) = 110$; not a very good approximation!
$L(4.1) = 56.5$, $f(4.1) = 56.621$; a better approximation.
$L(4.01) = 52.45$, $f(4.01) = 52.451201$; a very close approximation.

7. (a) The x-coordinate of the landing spot is $x = 25\cos\theta$. Our goal is to find $\mathrm{d}x$ when $\theta = \frac{\pi}{6}$ and the potential error is $\mathrm{d}\theta = \frac{\pi}{12}$. We have

$$\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta = -\sin\frac{\pi}{6}\cdot\frac{\pi}{12} = -\frac{1}{2}\cdot\frac{\pi}{12} = -\frac{\pi}{24}$$

(b) Similarly to (a), the y-coordinate of the landing spot is $y = 25\sin\theta$. We try to find $\mathrm{d}y$:

$$\mathrm{d}y = \cos\theta\,\mathrm{d}\theta = \cos\frac{\pi}{6}\cdot\frac{\pi}{12} = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{12} = \frac{\pi\sqrt{3}}{24}$$

(c) The distance between the intended landing spot when $\theta = \frac{\pi}{6}$ and the maximum error landing spot when $\theta = \frac{\pi}{4}$ can be found by calculating the distance between the coordinates $(25\cos\frac{\pi}{6}, 25\sin\frac{\pi}{6})$ and $(25\cos\frac{\pi}{4}, 25\sin\frac{\pi}{4})$:

$$D = \sqrt{\left(25\cos\frac{\pi}{4} - 25\cos\frac{\pi}{6}\right)^2 + \left(25\sin\frac{\pi}{4} - 25\sin\frac{\pi}{6}\right)^2} = 6.526 \text{ meters}$$

Exercises 3.7 Solutions

1. We cannot use the derivative of e^x in the proof of the derivative of e^x; this is circular logic.

2. (a) This is an indeterminate form of type $\frac{0}{0}$, so we may use L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\frac{1}{2}$$

where we used the Fundamental Trig Limit for the last step.

(b) This is an indeterminate form of type $\frac{\infty}{\infty}$, so

$$\lim_{x\to\infty}\frac{\ln x}{\ln(1+x^2)}=\lim_{x\to\infty}\frac{1/x}{2x/(1+x^2)}=\lim_{x\to\infty}\frac{1+x^2}{2x^2}=\frac{1}{2}$$

In the last step, we may use L'Hôpital's Rule again or use the rules discussed in Chapter 1.

(c) This is an indeterminate form of type $\frac{\infty}{\infty}$, but applying L'Hôpital's Rule yields a more complex limit. Instead, factorize first before applying L'Hôpital's Rule:

$$\lim_{x\to-\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}=\lim_{x\to-\infty}\frac{e^x(1+e^{-2x})}{e^x(1-e^{-2x})}=\lim_{x\to-\infty}\frac{1+e^{-2x}}{1-e^{-2x}}=\lim_{x\to-\infty}\frac{-2e^{-2x}}{2e^{-2x}}=-1$$

Beware that the situation changes if $x\to\infty$. In this case, we do *not* have an indeterminate form.

(d) This is an indeterminate form of type $\frac{0}{0}$, so we may use L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{x+\tan x}{x}=\lim_{x\to 0}\frac{1+\sec^2 x}{1}=\frac{1+1}{1}=2$$

Alternatively, we may split the fraction into two parts and use the Fundamental Trig Limit to evaluate the second fraciton.

(e) This is an indeterminate form of type $\frac{0}{0}$, so we may use L'Hôpital's Rule:

$$\lim_{\theta\to\frac{\pi}{2}}\frac{\theta-\pi/2}{\cos\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{1}{-\sin\theta}=-1$$

(f) This is an indeterminate product; use a reciprocal identity to rewrite the limit as an indeterminate form of type $\frac{\infty}{\infty}$:

$$\lim_{x\to 0^+}\ln x\cdot\sin x=\lim_{x\to 0^+}\frac{\ln x}{\csc x}=\lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x}=-\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\tan x=0$$

Notice the emergence of the Fundamental Trig Limit again.

3. This is an indeterminate power, which we can evaluate by using the discussed method:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^{\lim_{n\to 0} n\ln(1+\frac{1}{n})}$$

For this, we need

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln(1+n^{-1})}{n^{-1}} = \lim_{n\to\infty}\frac{-n^{-2}/(1+n^{-1})}{-n^{-2}} = \lim_{n\to\infty}\frac{1}{1+n^{-1}} = 1$$

Where the third equality is an application of L'Hôpital's Rule. Therefore our limit is $e^1 = e$.

4. (a) $g(0) \approx 6 + \frac{4}{5}(0+1) = \frac{34}{5}$. There is not enough information to determine whether this approximation is an overestimate or underestimate, since we have no information about the values of $g''(x)$ between $x = 0$ and $x = 1$.

   (b) We have

$$p'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x) \Longrightarrow p'(-1) = f'(-1)\cdot g(-1) + f(-1)\cdot g'(-1)$$
$$= \frac{4}{5}\cdot 6 + 6\cdot\frac{4}{5} = \frac{48}{5}$$

   The values of $f(-1)$ and $g(-1)$ are given. We found $f'(-1)$ and $g(-1)$ by noticing that the slope of the tangent line to f and g at $x = -1$ is $\frac{4}{5}$.

   (c) If $\lim\limits_{x\to -1} f(x)$ can be evaluated using L'Hôpital's Rule, that means the limit must be an indeterminate form. By inspection, when evaluated at $x = -1$, the numerator is 0, meaning the denominator must also be 0:

$$(h(-1))^5 - 32 = 0 \Longrightarrow h(-1) = 2$$

   Now actually apply L'Hôpital's Rule to the limit:

$$\lim_{x\to -1}\frac{1+x^3}{(h(x))^5 - 32} = \lim_{x\to -1}\frac{3x^2}{5(h(x))^4\cdot h'(x)} = 6$$

   We know the limit of f as $x \to -1$ must be 6 since f is twice-differentiable and thus continuous. If f is continuous, we must have $f(-1) = \lim\limits_{x\to -1} f(x) = 6$. Now

$$\frac{3(-1)^2}{5(h(-1))^4\cdot h'(-1)} = \frac{3}{5(2)^4\cdot h'(-1)} = 6 \Longrightarrow h'(-1) = \frac{1}{160}$$

   (d) If $h(1) = \sqrt[5]{33}$, then

$$f(1) = \frac{1+1^3}{(h(1))^5 - 32} = \frac{2}{33-32} = 2 \Longrightarrow \frac{f(1) - f(-1)}{1-(-1)} = \frac{2-6}{2} = -2$$

   Since f is continuous and differentiable, by the Mean Value Theorem, there must exist a c for $-1 < c < 1$ such that $f'(c) = -2$.